THE UNIVERSITY OF ALBERTA

THE NUMERICAL SOLUTION OF FUNCTIONAL EQUATIONS ARISING IN STOCHASTIC LEARNING MODELS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

by

GARRY SEYMOR MARLISS

EDMONTON, ALBERTA

OCTOBER, 1963

ABSTRACT

This thesis presents a study of the limiting distributions of response probabilities arising from two stochastic learning models derived by Bush and Mosteller [1]. The limiting distributions of the Experimenter-Controlled Model satisfy a functional equation which has proved to be extremely valuable in determining properties of, and solutions for the distribution functions. Under certain restrictions on its parameters, the functional equation yields a direct construction of purely singular Cantor-like functions. Under other conditions, absolutely continuous functions may be constructed from the functional equation. This leads to a sufficiency condition on the parameters for absolutely continuous distribution functions. Necessary and sufficient conditions for the limiting distribution to be the uniform distribution have also been found.

The limiting distributions of the Subject-Controlled Model satisfy a Volterra-like functional integral equation. Due to the complexity of this equation, its value in deriving useful results has been rather limited. However, it does display some properties of the limiting distributions in restricted cases.

The last half of the thesis is concerned with methods of calculating approximations to the limiting distributions by numerical solutions of the distribution functional equations. An error analysis of the procedures is impossible at this stage, but comparing the numerical solutions with known results in special cases indicates that the numerical procedure is sufficiently accurate for most practical purposes.

ACKNOWLEDGEMENT

My thanks to Dr. John McNamee for his understanding guidance and kindness extending over many years. To Dr. John Ross McGregor, my gratitude for suggesting the topic and for his constant help and encouragement during the preparation of this thesis.

My thanks also to the staff of the Computing Center of the University of Alberta for the use of their facilities.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Page

CHAPTER I

INTRODUCTION

Since 1949 a number of mathematicians and psychologists have been working on the formulation of mathematical models to describe learning processes. Thus far, these models have been restricted to the rather special situations of psychological learning experiments because of the simplifications imposed by experimental controls.

The original purpose for deriving learning models was to present a mathematical framework for analyzing statistical data from psychological experiments. Beside being of practical value for applied statistics, these models have given rise to a number of mathematically interesting problems. For example, some new problems in statistical estimation were opened up and also some new problems in the solution of functional equations have arisen. In the future, learning models may lead to new insights into the processes of learning, and so give rise to new and better learning theories.

The basic assumption of all present learning models is that the possible responses of a subject can be considered as related to the stimulus elements of the environment by a set of response probabilities. As a subject undergoes a number of experimental trials, its responses may be expected to vary as learning progresses. This change of behaviour is taken as a reflection of changing response probabilities. Existing learning models are the results of attempts to describe the changes of response probabilities in terms of mathematical transformations and to deduce the consequences of these transformations.

One of the earlier models is described by Bush and Mosteller in their book Stochastic Models for Learning [1]. The authors begin by defining learning as " ... any systematic change in behaviour ... ". Furthermore, learning is said to be complete when some type of stability of behaviour is attained. Using this operational definition of learning gives an objective basis for testing the model.

The general Bush and Mosteller learning model may be qualitatively described in terms of the following experiment of the reward training of rats in a T-maze. A T-shaped maze is set up with a food dish at each of the two ends of the cross-bar. Each experimental trial consists of setting a hungry rat at the starting point, the base of the T, and allowing it to run down the maze to the choice point where it must turn either to the right or to the left. Depending upon the type of experiment, the rat may find food or an empty dish.

The mathematical abstraction of the learning process is constructed from two considerations. Suppose that at the n th trial a given rat has a probability p_n of turning right and a probability $q_n = 1-p_n$ of turning left. If the rat turns right and finds food, then the probability of it turning right on the succeeding trial will be increased. Alternatively, if the rat turns right and is unrewarded, then p_{n+1} will be less than p_n. Bush and Mosteller assume p_{n+1} to be a linear function of p_n such that

$$p_{n+1} = a + bp_n , \tag{1.1}$$

where a and b are parameters depending upon the individual rat and whether the rat was rewarded or not on that trial.

From this general model of varying probabilities, a number of special models have been constructed to represent special learning experiments. The problems to be considered in this thesis arise from the "Experimenter-Controlled Event Model" and the "Subject-Controlled Event Model".

The Experimenter-Controlled Model may be illustrated as a variant of the T-maze experiment in which the experimenter has control over the frequencies of rewards and non-rewards that the rat recieves. In this learning situation the experimenter places food in the right dish on a fixed proportion, π_1 , of the trials on which the rat turns right. Similarly, he places food in the left dish on some other fixed proportion, π_2 , of the trials on which the rat turns left. On any given trial, the decision to reward or not to reward the rat is made by some suitable random procedure. The rewarding proportions are chosen such that $\pi_1 + \pi_2 = 1$.

The Subject-Controlled Model is another variant of the general learning model, but in this case the reward or non-reward is wholly dependent upon the subject's behaviour. In terms of the T-maze, the experiment is run with food placed on the same side at each trial. If the rat chooses the reward side, it receives food, otherwise it remains hungry. Thus the outcome of each trial is completely subject to the rat's choice.

Consider the effect of running a large number of initially identical rats through a series of trials of a T-maze experiment. Before the first trial all the rats possess the same probability p_o of turning right. At the end of the first trial the response probabilities of each rat will have

been modified depending upon the direction the rat chose and whether or not it had been rewarded. Thus the response probabilities of the rats will now be distributed amongst several values. At the end of the second trial the response probabilities will again have been modified. The value of p_2 will depend upon the value of p_1 as well as the choice and outcome of trial 2. Thus for each value of p_1 there will be several possible values of p_2. As the number of trials increases, the number of possible values of the response probability increases geometrically so that a series of distributions of response probabilities is generated. It is this set of distributions which is of value in applied statistics. Of particular interest is the limiting distribution of response probabilities as n becomes very large since it indicates the ultimate behaviour of the rats undergoing the learning experiment.

In the particular cases of the Experimenter-Controlled and Subject-Controlled Models, the general solutions for the limiting distribution functions of response probabilities are unknown. However, it will be shown that these functions satisfy certain functional equations with appropriate auxiliary conditions. Unfortunately there are no known methods for finding the general solutions of such functional equations, but these equations are useful in determining some of the properties of the limiting distributions, and in some special cases, explicit solutions are obtained.

The following chapters of this thesis are concerned with the properties of and the solutions for the limiting distributions of response probabilities. In chapters II and III, the Experimenter- and Subject-Controlled Models respectively are derived. From these, the functional equations which are satisfied by the characteristic functions of the limiting

distributions are found. The characteristic equations are then inverted to yield the functional equations satisfied by the distribution functions. The remainder of Chapters II and III are devoted to finding some of the properties and solutions of the functional equations. Chapters IV and V are concerned with the problem of tabulating numerical approximations to the limiting distributions by numerical solutions of the functional equations. These chapters also include some tables of the numerical solutions for selected cases. The final chapter summarizes the conclusions of the previous chapters.

CHAPTER II

THE EXPERIMENTER-CONTROLLED MODEL

§ 2.1 Introduction

In this chapter the Experimenter-Controlled Model for a learning experiment having two possible responses is treated in detail. The mathematical formulation of the basic model is derived using the assumption that learning can be considered as a process of systematic changes of responses to given stimuli. Since no two subjects can be expected to learn at exactly the same rate, there are many possible values of the response probabilities at the n th trial of the experiment. It is shown that the moment generating function of this distribution of response probabilities satisfies a certain recurrence relation. As the number of trials becomes large, the distribution of response probabilities tends to a limiting distribution whose moment generating function satisfies a functional equation.

The properties of this limiting distribution will be the major concern of this chapter. In order to examine this distribution, the moment generating functional equation is transformed to a functional equation satisfied by the characteristic function. A further transformation is made to standardize the characteristic functional equation before it is inverted to yield a functional equation satisfied by the limiting distribution function. The known properties of the limiting distribution are reviewed and a number of additional properties and explicit solutions are found from the functional equation.

§ 2.2 Derivation of the Model

Consider a simple learning experiment in which each trial consists of presenting a subject with two mutually exclusive alternatives A_1 and A_2. The subject's choice of A_1 is called response R_1, and the choice of A_2 is called response R_2. Thus R_1 and R_2 must be mutually exclusive and exhaustive responses.

It is assumed that at each trial the subject has a definite probability of choosing each of the two alternatives. These probabilities are denoted $Pr(R_i)$, $(i = 1, 2)$. From the condition of mutual exclusiveness if

$$(2.1) \qquad Pr(R_1) = p ,$$

on a particular trial, then

$$(2.2) \qquad Pr(R_2) = 1 - p ,$$

on that same trial.

In order that the experiment be meaningful, it must be arranged so that each time a response occurs, an outcome follows; let O_1 and O_2 be exhaustive and exclusive outcomes. From all these considerations the following events can be defined:

$$(2.3) \qquad \begin{array}{llll} E_1' : R_1 \text{ followed by } O_1 , & E_3' : R_2 \text{ followed by } O_1 , \\ E_2' : R_1 \text{ followed by } O_2 , & E_4' : R_2 \text{ followed by } O_2 . \end{array}$$

Assuming that the events E_1' and E_4' and the events E_2' and E_3' are equivalent, one may define the events

$$(2.4) \qquad \begin{aligned} E_1 &: \quad E_1' \cup E_4' , \\ E_2 &: \quad E_2' \cup E_3' . \end{aligned}$$

Under the above conditions, the new events E_1 and E_2 are exclusive and exhaustive.

The basic axiom of the model is that the occurrence of E_j $(j = 1, 2)$ on any trial modifies $Pr(R_1)$ for the succeeding trial. In the interests of simplicity, a linear transformation was chosen. Thus if $Pr(R_1) = p$ on the n^{th} trial, then the value of $Pr(R_1)$ on the $n+1^{st}$ trial is given by applying one of the linear operators Q_1 or Q_2 defined by

$$(2.5) \qquad Q_1 p = a_1 + \alpha_1 p ,$$

$$(2.6) \qquad Q_2 p = a_2 + \alpha_2 p ;$$

where a_i and α_i $(i = 1, 2)$ are parameters satisfying the conditions

$$(2.7) \qquad \begin{aligned} 0 \leq a_i &\leq 1 , \\ 0 \leq \alpha_i &\leq 1 - a_i . \end{aligned}$$

The choice of the operator is determined by the occurrence of event E_1 or E_2.

In experimenter-controlled learning experiments, the experimenter arranges that on a fixed proportion, π_1, of the trials on which R_1 occurs, O_1 follows. On the remaining trials that R_1 occurs, O_2 follows. From the conditions (2.3) and (2.4), this means that π_1 is the probability that operator Q_1 is applied, and π_2 the probability that Q_2 is applied.

Thus

$$\begin{aligned} Pr(E_1) &= \pi_1 , \\ Pr(E_2) &= \pi_2 . \end{aligned} \tag{2.8}$$

If the subject has some initial response probability $Pr(R_1) = p_o$, then the possible values of $Pr(R_1)$ throughout the first few trials will be generated as shown in Figure 2.1.

Since there are two possible events at each trial, at the n^{th} trial there must be 2^n possible values of $Pr(R_1)$. Let these values be denoted by $p_{m,n}$ $(m=1,2,3,\ldots,2^n ; n=1,2,3,\ldots)$. Similarly, let the corresponding probabilities of the occurrence of $p_{m,n}$ be

$$P_{m,n} = Pr(p_{m,n}) . \tag{2.9}$$

Thus if $Pr(R_1) = p_{m,n}$ on trial n, then on trial $n+1$ the possible response probabilities are:

$$\begin{aligned} Q_1 p_{m,n} &= a_1 + \alpha_1 p_{m,n} \quad \text{with probability } \pi_1 P_{m,n} , \\ Q_2 p_{m,n} &= a_2 + \alpha_2 p_{m,n} \quad \text{with probability } \pi_2 P_{m,n} . \end{aligned} \tag{2.10}$$

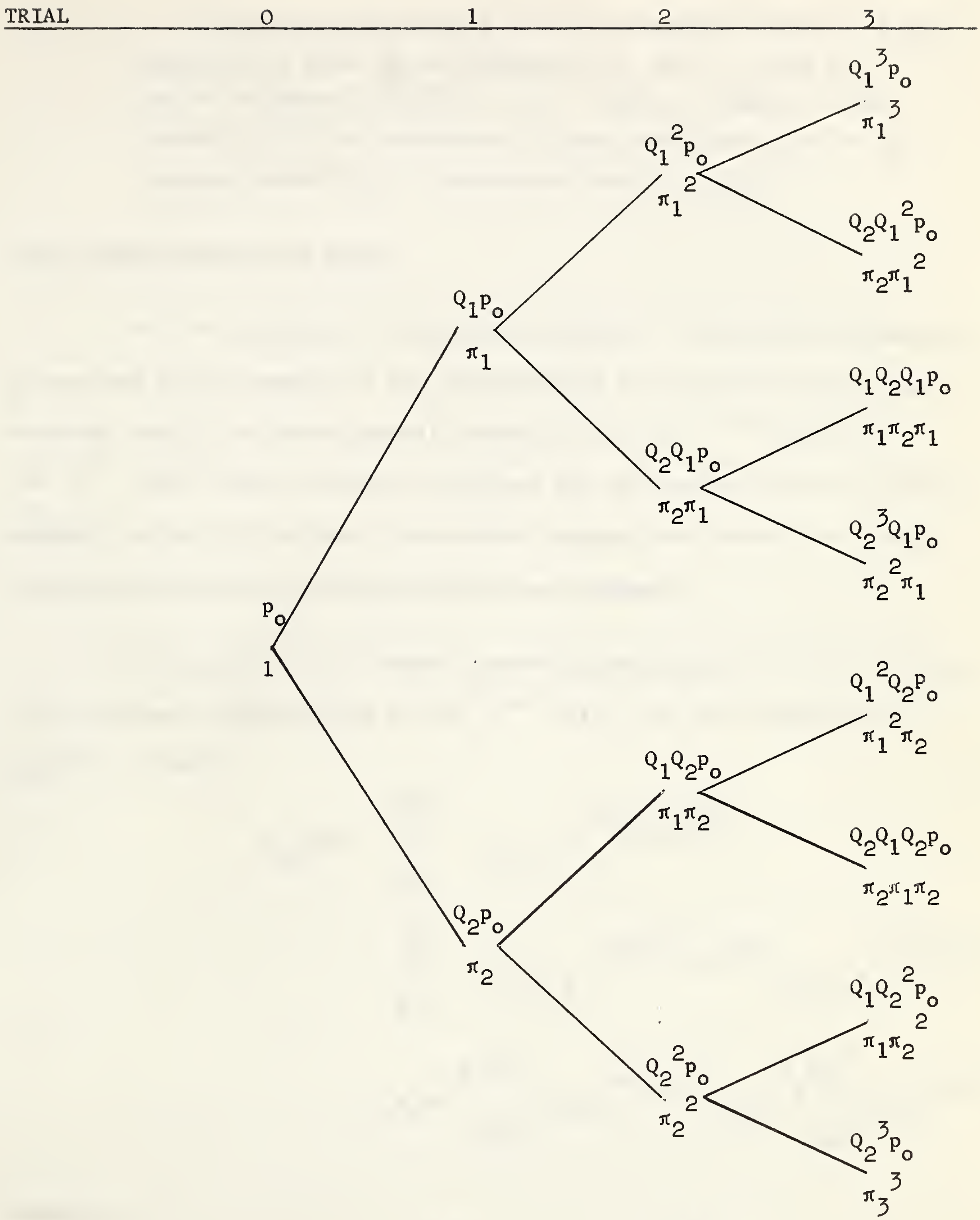

Figure 1 : Possible ways that the response probablity $P_r(R_1)$ can vary throughout the first three trials, given an initial value p_o.

The upper entry of each pair is a possible value of $Pr(R_1)$ expressed in terms of the operators Q_1 and Q_2 and the initial response probability p_o. The lower number is the probability of the occurrence of that particular value of the response probability. (Reproduced from [1] p.70).

§ 2.3 Properties of the Model

For the purposes of statistical analysis, much useful information is provided by the moments of the distributions of response probabilities. Although there is no known general expression for the m^{th} moment at the n^{th} trial, some recurrence relations for the moments have been found. McGregor and Hui [2] derived a recurrence relation satisfied by the moment generating functions by using the following argument.

Let $m_n(\theta)$ be the moment generating function of the distribution of the response probabilities at the n^{th} trial. By the definition of $m_{n+1}(\theta)$ we have

$$m_{n+1}(\theta) = \sum_{\ell=1}^{2^{n+1}} P_{\ell,n+1} \, e^{(p_{\ell,n+1}\theta)} ,$$

$$= \sum_{\ell=1}^{2^n} \left\{ \pi_1 P_{\ell,n} \, e^{(a_1+\alpha_1 p_{\ell,n})\theta} + \pi_2 P_{\ell,n} e^{(a_2+\alpha_2 p_{\ell,n})\theta} \right\}$$

$$= \pi_1 e^{a_1\theta} \sum_{\ell=1}^{2^n} P_{\ell,n} \, e^{\alpha_1 p_{\ell,n}\theta} + \pi_2 e^{a_2\theta} \sum_{\ell=1}^{2^n} P_{\ell,n} e^{\alpha_2 p_{\ell,n}\theta} .$$

Therefore

(2.11) $$m_{n+1}(\theta) = \pi_1 e^{a_1\theta} \, m_n(\alpha_1\theta) + \pi_2 e^{a_2\theta} \, m_n(\alpha_2\theta) .$$

Obviously the moment generating function of the distribution of response probabilities at the $n+1$ trial is dependent on all the moment generating functions of the previous trials.

Of particular importance to psychologists is the behaviour of the subject when complete learning has been attained. Karlin [3] opened the way to this study by proving that the limiting distributions exist as the number of trials tends to infinity. However, relatively little has been published about the properties of these limiting distributions. It is this problem which will be the concern of the remainder of this chapter.

The moment generating function of the limiting distribution of response probabilities satisfies the limiting form of equation (2.11). Thus if $\lim_{n\to\infty} m_n(\theta) = m(\theta)$, the above recurrence relation reduces to the functional equation

$$(2.12) \qquad m(\theta) = \pi_1 e^{a_1\theta} m(\alpha_1\theta) + \pi_2 e^{a_2\theta} m(\alpha_2\theta) \quad .$$

We transform (2.12) into a functional equation satisfied by the characteristic function $\varphi_p(\theta)$ by replacing θ by $i\theta$ giving

$$(2.13) \qquad \varphi_p(\theta) = \pi_1 e^{ia_1\theta} \varphi_p(\alpha_1\theta) + \pi_2 e^{ia_2\theta} \varphi_p(\alpha_2\theta) \ .$$

It has been shown by Karlin in his Trapping Theorem ([1] p.98) that the limiting response probabilities are trapped in the interval bounded by λ_1 and λ_2 where

$$(2.14) \qquad \lambda_1 = a_1/(1-\alpha_1) \quad , \quad \lambda_2 = a_2/(1-\alpha_2) \quad .$$

Thus if $F(p)$ is the limiting distribution function of response probabilities, the Trapping Theorem implies the boundary conditions

$$(2.15)\qquad \begin{aligned} F(p) &= 0 \quad , \quad p \leq \min(\lambda_1, \lambda_2) \quad ; \\ F(p) &= 1 \quad , \quad p \geq \max(\lambda_1, \lambda_2) \quad . \end{aligned}$$

Substituting for the a_i $(i = 1,2)$ in equation (2.13), we obtain

$$(2.16)\qquad \varphi_p(\theta) = \pi_1 e^{i\lambda_1(1-\alpha_1)\theta} \varphi_p(\alpha_1\theta) + \pi_2 e^{i\lambda_2(1-\alpha_2)\theta} \varphi_p(\alpha_2\theta) \quad .$$

For the sake of convenience, a change of variable will be made to map the limiting variable p onto the interval $[-1, 1]$. Without loss of generality it can be assumed that $\lambda_2 > \lambda_1$ since interchanging λ_1 with λ_2 and α_1 with α_2 and π_1 with π_2 in equation (2.16) does not change the equation. The required transformation is

$$(2.17)\qquad x = ((\lambda_1 + \lambda_2) - 2p)/(\lambda_1 - \lambda_2) \quad , \quad (\lambda_2 > \lambda_1) \quad .$$

By definition, $\varphi_p(\theta) = E(e^{i\theta p})$, where $E(e^{i\theta p})$ is the expectation of $e^{i\theta p}$. Making the transformation (2.17) then yields

$$\begin{aligned} \varphi_x(\theta) = E(e^{i\theta x}) &= E(e^{i\theta[((\lambda_1 + \lambda_2) - 2p)/(\lambda_1 - \lambda_2)]} \\ &= e^{i\theta(\lambda_1 + \lambda_2)/(\lambda_1 - \lambda_2)} \varphi_p(2\theta/(\lambda_2 - \lambda_1)) \quad . \end{aligned}$$

Putting $x = 2\theta/(\lambda_2 - \lambda_1)$, we get

$$\varphi_x(\tfrac{1}{2}s(\lambda_2 - \lambda_1)) = e^{-is(\lambda_1 + \lambda_2)/2} \varphi_p(s) \quad .$$

Thus

$$(2.18)\qquad \begin{aligned} \varphi_p(s) &= e^{is(\lambda_1 + \lambda_2)/2}\,\varphi_x(\tfrac{1}{2}s(\lambda_2 - \lambda_1)) \quad ; \\ \varphi_p(\alpha s) &= e^{i\alpha s(\lambda_1 + \lambda_2)/2}\,\varphi_x(\tfrac{1}{2}\alpha s(\lambda_2 - \lambda_1)) \;. \end{aligned}$$

Applying (2.18) to (2.16) and dropping the subscript x on $\varphi_x(\theta)$,

$$\begin{aligned} e^{\frac{1}{2}i\theta(\lambda_1+\lambda_2)}\varphi\,(\tfrac{1}{2}\theta(\lambda_2-\lambda_1)) &= \pi_1\, e^{i\theta[\lambda_1(1-\alpha_1)+\frac{1}{2}\alpha_1(\lambda_1+\lambda_2)]}\varphi(\tfrac{1}{2}\alpha_1\theta(\lambda_2-\lambda_1)) \\ &+ \pi_2\, e^{i\theta[\lambda_2(1-\alpha_2)+\frac{1}{2}\alpha_2(\lambda_1+\lambda_2)]}\varphi\,(\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)) \;, \end{aligned}$$

or,

$$\begin{aligned} \varphi(\tfrac{1}{2}\theta(\lambda_2-\lambda_1)) &= \pi_1\, e^{-\theta[\lambda_1(1-\alpha_1)-\frac{1}{2}(1-\alpha_1)(\lambda_1+\lambda_2)]}\varphi(\tfrac{1}{2}\alpha_1\theta(\lambda_2-\lambda_1)) \\ &+ \pi_2\, e^{i\theta[\lambda_2(1-\alpha_2)-\frac{1}{2}(1-\alpha_1)(\lambda_1+\lambda_2)]}\varphi(\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)) \;. \end{aligned}$$

Put $u = \frac{1}{2}(\lambda_2-\lambda_1)\varphi$

$$(2.19)\qquad \varphi(u) = \pi_1\, e^{-iu(1-\alpha_1)}\varphi(\alpha_1 u) + \pi_2\, e^{iu(1-\alpha_2)}\varphi(\alpha_2 u)$$

Because (2.19) is completely independent of λ_1 and λ_2 , it is obvious that these parameters merely served to define a scale factor. The change of variable (2.17) transforms the old boundary conditions to

$$(2.20)\qquad \begin{aligned} F(x) &= 0 \;, \quad x \le -1 \;; \\ F(x) &= 1 \;, \quad x \ge 1 \;. \end{aligned}$$

A distribution function may be obtained by applying the fundamental inversion theorem to its characteristic function (c.f. Kendall [4] p. 94).

This inversion theorem states that

$$(2.21) \qquad F(x) - F(0) = 1/(2\pi) \int_{-\infty}^{\infty} \{\varphi(t)(1 - e^{-ixt})/ (it)\} \, dt \ ,$$

where $F(x)$ is the required distribution function corresponding to the characteristic function $\varphi(t)$.

Inverting equation (2.19) by the formula of (2.21), the left member becomes

$$(2.22) \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(t)(1-e^{-ixt})/(it)\} \, dt = F(x) - F(0) \ ;$$

the first term of the right member becomes

$$\frac{\pi_1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(\alpha_1 t) \, e^{-it(1-\alpha_1)} \, (1-e^{-ixt})/(it)\} \, dt$$

which may be written as

$$\frac{\pi_1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(\alpha_1 t)[-(1-e^{-it(1-\alpha_1)}) + (1-e^{-it(x+1-\alpha_1)})]/(it)\} \, dt \ ,$$

or, putting $u = \alpha_1 t$

$$(2.23) \qquad \frac{\pi_1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(u)[-(1-e^{-iu(1-\alpha_1)/\alpha_1}) + (1-e^{-iu(x+1-\alpha_1)/\alpha_1})]/(iu)\} du$$

$$= \pi_1[-F((1-\alpha_1)/\alpha_1) + F(0) + F((x+1-\alpha_1)/\alpha_1) - F(0)]$$

$$= \pi_1[F((x+1-\alpha_1)/\alpha_1) - F((1-\alpha_1)/\alpha_1)] \ ;$$

while the second term of the right member becomes

$$\frac{\pi_2}{2\pi}\int_{-\infty}^{\infty}\{\varphi(\alpha_2 t)\ e^{it(1-\alpha_2)}(1-e^{-ixt})/(it)\}\ dt$$

$$= \frac{\pi_2}{2\pi}\int_{-\infty}^{\infty}\{\varphi(\alpha_2 t)[-(1-e^{-it(\alpha_2-1)}) + (1-e^{-it(x-1+\alpha_2)})]/(it)\}\ dt\ ,$$

or, letting $v = \alpha_2 t$

$$(2.24)\qquad \frac{\pi_2}{2\pi}\int_{-\infty}^{\infty}\{\varphi(v)[-(1-e^{-iv(\alpha_2-1)/\alpha_2}) + (1-e^{-iv(x-1+\alpha_2)/\alpha_2})]/(iv)\}\ dv$$

$$= \pi_2[F((x-1+\alpha_2)/\alpha_2) - F(\alpha_2-1)/\alpha_2)]\ .$$

Collecting the inverted terms (2.21), (2.22), (2.23) and (2.24) the result is

$$F(x)-F(0) = \pi_1[F((x+1-\alpha_1)/\alpha_1)-F((1-\alpha_1)/\alpha_1)]+\pi_2[F((x-1+\alpha_2)/\alpha_2) -F((\alpha_2-1)/\alpha_2)]\ .$$

Putting $x = -1$ and imposing the boundary condition (2.20), we obtain

$$-F(0) = -\pi_1 F((1-\alpha_1)/\alpha_1) - \pi_2 F((\alpha_2-1)/\alpha_2)\ .$$

Thus the transformed limiting distribution function satisfies the homogeneous functional equation

$$(2.25)\qquad F(x) = \pi_1 F((x+1-\alpha_1)/\alpha_1)+\pi_2 F((x-1+\alpha_2)/\alpha_2)\ .$$

The general solutions for the limiting distribution functions are as yet unknown, but the properties demonstrated by the few published results indicate that these functions exhibit rather unusual behaviour. Because most of these results were derived directly from the characteristic

functional equation (2.16) or from the properties of the operators Q_1 and Q_2 , explicit distribution functions were found in only a very few cases. Furthermore, the known results are generally restricted to equal alphas $(\alpha = \alpha_1 = \alpha_2)$ or equal pis $(\pi_1 = \pi_2 = \frac{1}{2})$.

The following review outlines the major contributions to the present knowledge of the properties of the limiting distributions of response probabilities. Jessen and Wintner [5] proved that in the case of equal alphas and equal pis, the distribution functions are either purely singular or absolutely continuous. Kershner and Wintner [6] showed that for $0 < \alpha < \frac{1}{2}$, the limiting distributions are purely singular and their frequency functions $[\frac{d}{dx} F(x)]$ are non-zero only on a set of points of Lebesgue measure zero. In particular, the frequency function for the case $\alpha = \frac{1}{3}$ is non-zero only on the Cantor Ternary Set. They also proved that for $\alpha = \frac{1}{2}$ and $\pi_1=\pi_2=\frac{1}{2}$, the distribution function is absolutely continuous and indeed, in this case, $F(x) = \frac{1}{2}(x+1)$, $(-1 \leq x \leq 1)$. Additional properties for the case $\alpha = \frac{1}{2}$ were derived by Kemeny and Snell [7]. These authors proved that for all $\pi_1 \neq \pi_2$, the limiting distributions are again purely singular. Karlin [3] generalized many of the above results. He was able to prove that the limiting distributions are either purely singular or absolutely continuous for all possible values of the parameters. Furthermore these distributions are purely singular for all $\alpha_1 + \alpha_2 < 1$.

For the remaining cases of $\alpha_1 + \alpha_2 > 1$, less is known. Wintner [8] showed that for the case of equal pis and alphas with $\alpha = 2^{-1/n}$ $(n = 1,2,3, \ldots)$, the limiting distribution functions are absolutely continuous

and even possess $n-1$ continuous derivatives. McGregor and Hui [2] demonstrated that in the equal pi and alpha case, for $\alpha = 1/\sqrt{2}$, the characteristic functional equation (2.19) could be written as the characteristic function of a convolution of uniform distributions over unequal intervals. Using this result, the distribution function was found to be composed of three piecewise continuous polynomials of degree two, Zidek [9] generalized this approach and showed that for $\pi_1=\pi_2=\frac{1}{2}$, and $\alpha = 2^{-1/n}$ $(n = 3,4,5, \ldots)$, the distribution functions consist of 2^n-1 piecewise continuous polynomials of degree n. Finally, in an early paper, Erdos [10] proved that for $\pi_1=\pi_2=\frac{1}{2}$, there exists a set of alphas in the interval $\frac{1}{2} < \alpha < 1$ which yield purely singular distributions. These values of α lie between the values which yield the absolutely continuous piecewise polynomials.

§ 2.4 Some Additional Properties of the Limiting Distributions

In the previous section it was shown that the distribution functions of the limiting distributions of the transformed response probabilities satisfy the functional system*

(2.26) $$F(x) = \pi_1 F((x+1-\alpha_1)/\alpha_1) + \pi_2 F((x-1+\alpha_2)/\alpha_2) \;;$$

(2.27) $$F(x) = 0 \;,\; x \leq -1 \;;\quad F(x) = 1 \;,\; x \geq 1 \;;$$

(2.28) $$\pi_1+\pi_2 = 1 \;,\quad 0 \leq \pi_1,\pi_2 \leq 1 \;,\quad 0 < \alpha_1 \;,\; \alpha_2 < 1 \;.$$

---

* Functional equation and auxiliary conditions.

<u>Theorem 2.1</u> For $\alpha_1 + \alpha_2 < 1$ the solutions of the functional system (2.26) - (2.28) are purely singular functions which are constant everywhere except at a non-denumerable set of points of Lebesgue measure zero.

<u>Proof:</u> The proof of this theorem is composed of three distinct parts. In the first part an algorithm for constructing the exact distribution functions is derived. This algorithm shows that the distribution functions take constant values on a set of closed intervals lying in the domain (-1, 1). The second part of the proof comprises Lemma 2.1 which shows that the set of intervals defined by the algorithm are disjoint for all $\alpha_1 + \alpha_2 \leq 1$. The third part of the proof starts with Lemma 2.2. In this lemma it is proved that the set of disjoint intervals covers the entire domain (-1, 1) except for a non-denumerable set of points of Lebesgue measure zero. Finally, the proof of the theorem (p. 27) follows from the fact that the distribution can be uniquely constructed.

In the open interval $-1 < x < 1$, the arguments of (2.26) are ordered such that $(x+1-\alpha_1)/\alpha_1 > x > (x-1+\alpha_2)/\alpha_2$. Since

$$(x+1-\alpha_1)/\alpha_1 \geq 1 \quad \text{when} \quad x \geq 2\alpha_1 - 1$$

and

$$(x-1+\alpha_2)/\alpha_2 \leq -1 \quad \text{when} \quad x \leq 1 - 2\alpha_2 ,$$

the boundary conditions (2.29) define a solution if an interval exists such that

$$1 - 2\alpha_2 > 2\alpha_1 - 1 \quad .$$

This condition is satisfied if and only if

$$\alpha_1 + \alpha_2 < 1 \quad .$$

Thus for any x in the interval

(2.29) $$2\alpha_1 - 1 \leq x \leq 1 - 2\alpha_2$$

we have

(2.30) $$F(x) = \pi_1 \quad , \qquad [2\alpha_1 - 1, 1 - 2\alpha_2] \quad .$$

From the above solution function, two new intervals and their corresponding solutions can be constructed. One of these intervals is defined by

$$2\alpha_1 - 1 \leq (x+1-\alpha_1)/\alpha_1 \leq 1 - 2\alpha_2 \quad .$$

Solving for x gives the interval

(2.31) $$\alpha_1(2\alpha_1-1) - (1-\alpha_1) \leq x \leq \alpha_1(1-2\alpha_2) - (1-\alpha_1) \quad .$$

The other interval is given by the condition

$$2\alpha_1 - 1 \leq (x-1+\alpha_2)/\alpha_2 \leq 1 - 2\alpha_2 \quad ,$$

which defines the interval

(2.32) $$\alpha_2(2\alpha_1-1) + (1-\alpha_2) \leq x \leq \alpha_2(1-2\alpha_2) + (1-\alpha_2) \quad .$$

The interval (2.31) lies to the left of (2.29) and is disjoint from it. The contrary assumption that

$$\alpha_1(1-2\alpha_2) - (1-\alpha_1) \geq 2\alpha_1 - 1$$

leads to the contradiction

$$-2\alpha_1\alpha_2 \geq 0 \quad , \text{ since } \quad (0 < \alpha_1 \ , \ \alpha_2 < 1) \quad .$$

Similarly the interval (2.32) lies to the right of (2.29)and is disjoint since the contrary assumption

$$1 - 2\alpha_2 \geq \alpha_2(2\alpha_1-1) + (1 - \alpha_2)$$

yields

$$0 \geq 2\alpha_1\alpha_2 \quad .$$

Thus the new intervals are disjoint for all acceptable values of α_1 and α_2.

The solutions over the new intervals are defined as follows:

i) when $x \in (2.31)$, $(x-1+\alpha_2)/\alpha_2 < -1$. Substituting into the functional equation (2.26) gives

(2.33) $$F(x) = \pi_1^{\;2} \quad , \quad [\alpha_1(2\alpha_1-1) - (1-\alpha_1), \alpha_1(1-2\alpha_2) - (1-\alpha_1)] \ .$$

ii) when $x \in (2.32)$, $(x+1-\alpha_1)/\alpha_1 > 1$ so the functional equation gives

(2.34) $$F(x) = \pi_1 + \pi_2\pi_1 \ , \ [\alpha_2(2\alpha_1-1) + (1-\alpha_2), \alpha_2(1-2\alpha_2) + (1-\alpha_2)] \ .$$

In general, if x_e is an endpoint of an interval, new endpoints x_e' are determined from x_e by

$$(x_e'+1-\alpha_1)/\alpha_1 = x_e \qquad \text{giving} \quad x_e' = \alpha_1 x_e-(1-\alpha_1) \quad ,$$

$$(x_e'-1+\alpha_2)/\alpha_2 = x_e \qquad \text{giving} \quad x_e' = \alpha_2 x_e+(1-\alpha_2) \quad .$$

For later convenience, define the two transformations

$$B_1 x = \alpha_1 x - (1-\alpha_1) \quad , \tag{2.35}$$

$$B_2 x = \alpha_2 x + (1-\alpha_2) \quad . \tag{2.36}$$

At any given stage, the operators B_1 and B_2 generate the new intervals by a mapping of the intervals determined at the previous stage. In particular, the new endpoints are given by the transformations $B_1 x_e$ and $B_2 x_e$, where $\{x_e\}$ represents known endpoints.

Applying the algorithm a third time gives four new intervals with their corresponding solutions.

$$F(x)=\pi_1^3 \quad , \quad [\alpha_1^2(2\alpha_1-1)-\alpha_1(1-\alpha_1)-(1-\alpha_1),\alpha_1^2(1-2\alpha_2)-\alpha_1(1-\alpha_1)-(1-\alpha_1)] \; ; \tag{2.37}$$

$$F(x)=\pi_1^2+\pi_1^2\pi_2 \quad , \quad [\alpha_1\alpha_2(2\alpha_1-1)+\alpha_1(1-\alpha_2)-(1-\alpha_1),\alpha_1\alpha_2(1-2\alpha_2)+\alpha_1(1-\alpha_2)-(1-\alpha_2)] \; ; \tag{2.38}$$

$$F(x)=\pi_1+\pi_2\pi_1^2 \quad , \quad [\alpha_2\alpha_1(2\alpha_1-1)-\alpha_2(1-\alpha_1)+(1-\alpha_2),\alpha_2\alpha_1(1-2\alpha_2)-\alpha_2(1-\alpha_1)+(1-\alpha_2)] \; ; \tag{2.39}$$

$$F(x)=\pi_1+\pi_1\pi_2+\pi_1\pi_2^2 \quad , \quad [\alpha_2^2(2\alpha_1-1)+\alpha_2(1-\alpha_2)+(1-\alpha_2),\alpha_2^2(1-2\alpha_2)+\alpha_2(1-\alpha_2)+(1-\alpha_2)] \; . \tag{2.40}$$

Since each interval at the n^{th} stage is defined by the transformation B_1 or B_2 applied to the endpoints of an interval found at the $n\text{-}1^{st}$ stage, there must be 2^{n-1} intervals to be found at the n^{th} application of the algorithm. This procedure for constructing the solution may be repeated as often as required to outline the distribution function to

the desired degree of accuracy, provided only that the intervals do not overlap.

Lemma 2.1: The intervals generated in the above algorithm for constructing the solutions of the functional system (2.26)-(2.28) for $\alpha_1+\alpha_2<1$ are disjoint.

Proof: The following properties of the transformations B_1 and B_2 are required for the proof.

i) $B_1x = \alpha_1x - (1-\alpha_1)$ and $B_2x = \alpha_2x+(1-\alpha_2)$ are linear functions of x. Thus $B_1x' > B_1x$ and $B_2x' > B_2x$ if and only if $x' > x$.

ii) $B_1x < x$ for all $x \in (-1, 1)$, since by assuming the contrary that

$$\alpha_1x - (1-\alpha_1) > x$$

gives $x < -1$, which is a contradiction. Furthermore, $B_1x \in (-1, 2\alpha_1-1)$ for all $x \in (-1, 1)$.

iii) $B_2x > x$ for all $x \in (-1, 1)$. Again the contrary assumption

$$\alpha_2x + (1-\alpha_2) < x$$

gives the contradiction

$$x > 1 .$$

In this case $B_2x \in (1-2\alpha_2, 1)$ for all $x \in (-1, 1)$.

Given an interval $(x,y) = I$, define B_1I to be the interval (B_1x, B_1y) and B_2I to be the interval (B_2x, B_2y).

At the first stage in constructing the solution, the functional equation and boundary conditions define the interval $(2\alpha_1-1,\ 1-2\alpha_2)$. Denote this interval by $I_{1,1}$. The remaining intervals at this stage, namely $(-1,\ 2\alpha_1-1)$ and $(1-2\alpha_2,\ 1)$ we shall call gaps and denote them by $G_{1,1}$ and $G_{1,2}$.

At the second stage, consider the effects of the transformations B_1 and B_2. B_1 maps the domain $(-1,\ 1)$ onto $(-1,\ 2\alpha_1-1)$. Thus $G_{1,1} \supseteq B_1[G_{1,1} \cup I_{1,1} \cup G_{1,2}]$. Denote

$$
\begin{aligned}
& B_1G_{1,1} && \text{by} && G_{2,1}\ , \\
& B_1I_{1,1} && \text{by} && I_{2,1}\ , \\
\text{and}\quad & B_1G_{1,2} && \text{by} && G_{2,2}\ .
\end{aligned}
$$

Since B_2 maps $(-1,\ 1)$ onto $(1-2\alpha_2,\ 1)$ we have $G_{1,2} \supseteq B_2[G_{1,1} \cup I_{1,1} \cup G_{1,2}]$. Denote

$$
\begin{aligned}
& B_2G_{1,1} && \text{by} && G_{2,3}\ , \\
& B_2I_{1,1} && \text{by} && I_{2,2}\ , \\
\text{and}\quad & B_2G_{1,2} && \text{by} && G_{2,4}\ .
\end{aligned}
$$

Thus $I_{2,1}$ and $I_{2,2}$ are bounded by $G_{2,i}(i=1,2,3,4)$.

The same argument also holds at the third stage. Since $B_1G_{1,1}$ maps onto $G_{2,1}$ then $B_1[G_{2,1} \cup I_{2,1} \cup G_{2,2}]$ must map onto $G_{2,1}$. Also $B_1[G_{2,3} \cup I_{2,2} \cup G_{2,4}]$ must map onto $G_{2,2}$. Thus the new intervals and gaps generated by B_1 at the third stage may be denoted as follows.

$$B_1G_{2,1} \quad \text{by} \quad G_{3,1} \quad ,$$

$$B_1I_{2,1} \quad \text{by} \quad I_{3,1} \quad ,$$

$$B_1G_{2,2} \quad \text{by} \quad G_{3,2} \quad ,$$

$$B_1G_{2,3} \quad \text{by} \quad G_{3,3} \quad ,$$

$$B_1I_{2,2} \quad \text{by} \quad I_{3,2} \quad ,$$

and $$B_1G_{2,4} \quad \text{by} \quad G_{3,4} \quad .$$

The new intervals generated by B_2 are located such that

$$G_{2,3} \supseteq B_2[G_{2,1} \cup I_{2,1} \cup G_{2,2}] \quad , \quad G_{2,4} \supseteq B_2[G_{2,3} \cup I_{2,2} \cup G_{2,4}] \; .$$

Continuing the notation, denote

$$B_2G_{2,1} \quad \text{by} \quad G_{3,5} \quad ,$$

$$B_2I_{2,1} \quad \text{by} \quad I_{3,3} \quad ,$$

$$B_2G_{2,3} \quad \text{by} \quad G_{3,6} \quad ,$$

$$B_2G_{2,3} \quad \text{by} \quad G_{3,7} \quad ,$$

$$B_2I_{2,2} \quad \text{by} \quad I_{3,4} \quad ,$$

and $$B_2G_{2,4} \quad \text{by} \quad G_{3,8} \quad .$$

From this scheme it is obvious that in the general case, any new interval will be bounded by two gaps.

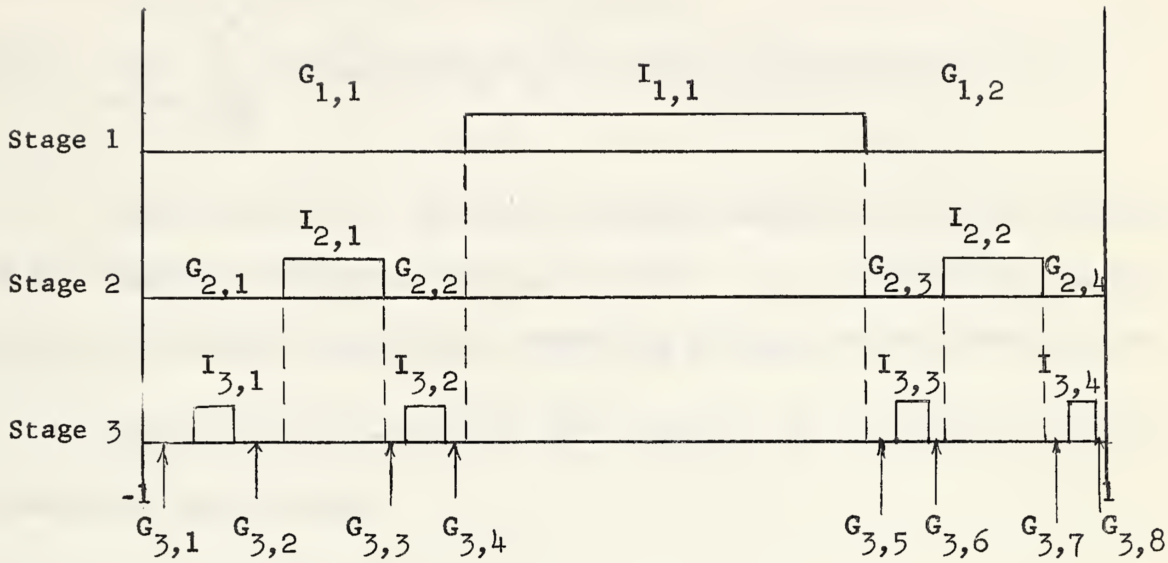

Figure 2. Formation of the Intervals of Solution in the first Three Stages

<u>Lemma 2.2</u>: In the limit as $n \to \infty$, the set of disjoint intervals $I_{m,i}$ $(m=1,2,3, \ldots, n;\ i=1,2, \ldots, 2^{m-1})$ covers the domain $(-1, 1)$ except for a non-denumerable set of points of Lebesgue measure zero.

<u>Proof</u>: Given an interval $[x_1, x_2]$ of length $\ell = x_2-x_1$, B_1 maps the interval onto the new interval $[\alpha_1 x_1-(1-\alpha_1), \alpha_1 x_2-(1-\alpha_1)]$ of length $\alpha_1(x_2-x_1) = \alpha_1\ell$. Similarly B_2 maps the given interval onto $[\alpha_2 x_1+(1-\alpha_2), \alpha_2 x_2+(1-\alpha_2)]$ of length $\alpha_2(x_2-x_1) = \alpha_2\ell$. Since both B_1 and B_2 map every interval $I_{m-1,i}$ to define the $I_{m,j}$, the sums of the lengths of $I_{m,j}$ $(j=1,2,\ldots, 2^{m-1})$ is $\alpha_1+\alpha_2$ times the sums of the lengths of $I_{m-1,i}$ $(i=1,2,\ldots,2^{m-2})$. Thus the sums of the lengths of $\{I_{m,i}\}$ over

m gives a geometric series with initial value $(1-2\alpha_2)-(2\alpha_1-1) = 2(1-\alpha_1-\alpha_2)$ and common ratios $(\alpha_1+\alpha_2)$. Summing the series

$$\lim_{m\to\infty} \sum_{n=0}^{m} 2(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)^n = 2(1-\alpha_1-\alpha_2)/[1-(\alpha_1+\alpha_2)] = 2 \quad . \tag{2.41}$$

This value is the sum of the Lebesgue measures of all the intervals. Thus the sequence of disjoint intervals covers $(-1, 1)$ except for the set of points of Lebesgue measure zero comprising the gaps of the limiting case.

The number of gaps at the n^{th} stage is 2^n . Thus in the limit, the number of gaps becomes

$$\lim_{N\to\infty} 2^N \quad .$$

In the limit N is of cardinality $\aleph_0$. A well known theorem of transfinite numbers states that

$$2^{\aleph_0} = c \tag{2.42}$$

where c is the cardinality of the continuum.

<u>Proof of Theorem 2.1</u>

The proof follows immediately from lemmas 2.1 and 2.2 and by noting that the solutions given by the algorithm are constant over the intervals $\{I_{n,i}\}$.

Thus for the case $\alpha_1+\alpha_2 < 1$ the limiting distributions of the transformed response probabilities are uniquely defined and can be constructed

to any desired degree of accuracy.

In the case $\alpha_1+\alpha_2=1$, the intervals $I_{n,i}$ of theorem 2.1 shrink to points, since $1-2\alpha_2 = 2\alpha_1-1$. However the algorithm for constructing the solution may be used to find the values of the function at a non-denumerable set of points. In addition to the construction, we have the following special result.

Theorem 2.2: Necessary and sufficient conditions that the limiting distribution be the uniform distribution are $\alpha_1 = \pi_1$, $\alpha_2 = \pi_2$, $(\pi_1+\pi_2=1)$.

Proof of Sufficiency.

The theorem will be proved by showing that the moments of the distribution for $\alpha_1=\pi_1$ and $\alpha_2=\pi_2$ are the moments of the uniform distribution. Before the main theorem can be proved, we again require an intermediate result.

Lemma 2.3: The characteristic functional equation (2.19)

$$\varphi(u) = \pi_1 e^{-iu(1-\alpha_1)} \varphi(\alpha_1 u) + \pi_2 e^{iu(1-\alpha_2)} \varphi(\alpha_2 u) ,$$

with $\varphi(0) = 1$ defines a unique set of moments V_m where

$$(i)^m V_m = \frac{d^m}{du^m} [\varphi(u)] .$$

Proof: Putting $u = 0$ and substituting $\varphi(0) = 1$ in the above functional equation merely reaffirms the identity

$$\pi_1 + \pi_2 = 1 .$$

Differentiating the functional equation with respect to u gives

$$\varphi'(u) = \alpha_1\pi_1 e^{-iu(1-\alpha_1)} \varphi'(\alpha_1 u) - \alpha_2\pi_2 e^{-iu(1-\alpha_2)} \varphi'(\alpha_2 u)$$

$$+ \; i[-\pi_1(1-\alpha_1)e^{-iu(1-\alpha_1)} \varphi(\alpha_1 u) + \pi_2(1-\alpha_2)e^{iu(1-\alpha_2)} \varphi(\alpha_2 u)] \; .$$

Therefore

$$\varphi'(0) = i[-\pi_1(1-\alpha_1) + \pi_2(1-\alpha_2)]/[1-\alpha_1\pi_1-\alpha_2\pi_2] \; .$$

$\varphi''(0)$ may be evaluated by differentiating the functional equation twice and substituting for $\varphi(0)$ and $\varphi'(0)$. Repeating this procedure, all the terms $\frac{d^m}{du^m} \varphi(u)]_{u=o}$ $(m=1,2,3,\ldots)$ can be evaluated in turn. Thus the moments are uniquely defined by (2.19).

<u>Proof of Sufficiency:</u>

The theorem will be proved by showing that the moments of the distribution for $\alpha_1 = \pi_1$ and $\alpha_2 = \pi_2$ are the moments of the uniform distribution.

The characteristic function of the uniform distribution over $(-1, 1)$ is

$$\varphi(t) = \tfrac{1}{2}\int_{-1}^{1} e^{ixt} \, dt$$

$$= \tfrac{1}{2} \, [e^{it} - e^{-it}]/(it)$$

(2.43) $$\varphi(t) = \frac{\sin (t)}{t}$$

Thus the moments V_m are given by

(2.44) $$V_m = [1 + (-1)^m]/[2(m+1)] \quad .$$

Bush and Mosteller ([1] p.88) give the following recurrence formula for the moments of the distribution of response probabilities at the $n+1^{st}$ trial as

$$V_{m,n+1} = \pi_1 \sum_{u=o}^{m} \binom{m}{u} a_1^{m-u} \alpha_1^{u} V_{u,n} + \pi_2 \sum_{u=o}^{m} \binom{m}{u} a_2^{m-u} \alpha_2^{u} V_{u,n} \quad .$$

In the limiting case, the recurrence formula becomes

$$V_m = \pi_1 \sum_{u=o}^{m} \binom{m}{u}\, a_1^{m-u} \alpha_1^{u} V_m + \pi_2 \sum_{u=o}^{m} \binom{m}{u}\, a_2^{m-u} \alpha_2^{u} V_m \quad ,$$

where $\quad V_m = \lim\limits_{n \to \infty} \; V_{m,n} \quad .$

Solving for V_m gives

$$V_m(1-\pi_1\alpha_1^{m}-\pi_2\alpha_2^{m}) = \pi_1 \sum_{u=o}^{m-1} \binom{m}{u}\, a_1^{m-u}\, \alpha_1^{u} V_u + \pi_2 \sum_{u=o}^{m-1} \binom{m}{u}\, a_2^{m-u} \alpha_2^{u} V_u \quad .$$

Transforming the above equation so that it defines the moments for the transformed distribution over $(-1, 1)$ and putting

$$\alpha_1 = \alpha \quad , \quad \alpha_2 = 1 - \alpha \quad ,$$

$$\pi_1 = \alpha \quad , \quad \pi_2 = 1 - \alpha \quad ,$$

yields

(2.45) $$V_m(1-\alpha^{m+1}-(1-\alpha)^{m+1}) = \alpha \sum_{u=o}^{m-1} \binom{m}{u}\, (-1)^{m-u} \alpha^{u} V_u + (1-\alpha) \sum_{u=o}^{m-1} \binom{m}{u} \alpha^{m-u} (1-\alpha)^{u} V_u \quad .$$

By definition $V_o = 1$.

The induction hypothesis is $V_m = [1+(-1)^m]/[2(m+1)]$. For $m=1$, (2.45) gives

$$V_1[1-\alpha^2-(1-\alpha)^2] = -\alpha(1-\alpha)V_o + (1-\alpha)\alpha V_o = 0 .$$

Therefore $V_1 = 0$.

Assuming the induction hypothesis to hold for $n \leq m-1$, (2.45) yields

$$V_m[1-\alpha^{m+1} -(1-\alpha)^{m+1}] = \tfrac{1}{2}\sum_{u=o}^{m-1} \binom{m}{u}(-1)^{m-u}(1-\alpha)^{m-u}\alpha_1^{u+1}[1+(-1)^u]/(u+1)$$

$$+ \tfrac{1}{2}\sum_{u=o}^{n-1} \binom{m}{u}\alpha^{m-u}(1-\alpha)^u[1+(-1)^u]/(u+1) ,$$

$$= \frac{1}{2(m+1)}\sum_{u=o}^{m-1} \binom{m+1}{u+1}(-1)^{m-u}(1-\alpha)^{m-u}\alpha^{u+1}(1+(-1)^u)$$

$$+ \frac{1}{2(m+1)}\sum_{u=o}^{m-1} \binom{m+1}{u+1} \alpha^{m-u}(1-\alpha)^{u+1}[1+(-1)^u] .$$

Thus

$$V_m[1-\alpha^{m+1}-(1-\alpha)^{m+1}] = \frac{1}{2(m+1)}\sum_{\nu=1}^{m}(1+(-1)^{\nu-1})\binom{m+1}{\nu}[(-1)^{m+1-\nu}(1-\alpha)^{m+1-\nu} \times \alpha^{\nu}+\alpha^{m+1-\nu}(1-\alpha)^{\nu}] . \tag{2.46}$$

Case I : $m = 2k+1$ $(k=0,1,2,\ldots)$.

i) When ν is even, the corresponding term of the sum vanishes.

ii) When ν is odd, say $\nu = 2\ell +1$, $(\ell=0,1,2,\ldots)$, the summand is

$$\binom{2k+2}{2\ell+1}[-(1-\alpha)^{2(k-\ell)+1}\alpha^{2\ell+1}+\alpha^{2(k-\ell)+1}(1-\alpha)^{2\ell+1}] .$$

Now let $\nu = 2(k+1)-2\ell-1 = 2(k-\ell)+1$. The summand becomes

$$\binom{2k+2}{2(k+1)-2\ell-1}[-(1-\alpha)^{2\ell+1}\alpha^{2(k-\ell)+1}+\alpha^{2\ell+1}(1-\alpha)^{2(k-\ell)+1}] .$$

In this case the terms of the sum cancel out in pairs so $V_{2k+1}=0$.

Case II : $m = 2k$ $(k=0,1,2,\ldots)$.

i) When ν is even, the corresponding term of the sum vanishes.

ii) When ν is odd, say $\nu = 2\ell+1$ $(\ell=0,1,2,\ldots)$, equation (2.46) becomes

$$V_{2k}[1-\alpha^{2k+1}-(1-\alpha)^{2k+1}] = \frac{1}{2k+1}\sum_{\ell=0}^{k-1}\binom{2k+1}{2\ell+1}[(1-\alpha)^{2(k-\ell)}\alpha^{2\ell+1}+\alpha^{2(k-\ell)} \times (1-\alpha)^{2\ell+1}] .$$

The sum on the right hand side is simply the binomial expansion

$$(1-\alpha+\alpha)^{2k+1} - \alpha^{2k+1} - (1-\alpha)^{2k+1} = 1-\alpha^{2k+1} - (1-\alpha)^{2k+1} .$$

Therefore

$$V_{2k} = \frac{1}{2k+1}$$

and sufficiency is proved.

Proof of Necessity:

The zeroth, first and second moments impose the conditions

(2.47) $$\pi_1 + \pi_2 = 1$$

(2.48) $$\pi_1(1-\alpha_1) = \pi_2(1-\alpha_2)$$

(2.49) $$3[\pi_1(1-\alpha_1)^2 + \pi_2(1-\alpha_2)^2] = 1 - \pi_1\alpha_1^2 - \pi_2\alpha_2^2 .$$

From (2.48)

(2.50) $$(1-\alpha_2) = \pi_2(1-\alpha_1)/\pi_2 ,$$

whence

(2.51) $$\alpha_2 = 1 - \pi_1(1-\alpha_1)/\pi_2 .$$

Substituting for α_2 , $1-\alpha_2$ and π_2 in (2.49)

$$3[\pi_1(1-\alpha_1)^2+(1-\pi_1)\pi_1^2(1-\alpha_1)^2/(1-\pi_1)^2] = 1-\pi_1\alpha_1^2 -(1-\pi_1)[1-\pi_1(1-\alpha_1)/(1-\pi_1)]^2;$$

$$3\pi_1(1-\alpha_1)^2[1+\pi_1/(1-\pi_1)] = 1-\pi_1\alpha_1^2 - (1-\pi_1) + 2\pi_1(1-\alpha_1) - \pi_1^2(1-\alpha_1)^2/(1-\pi_1)$$

$$= \pi_1(1-\alpha_1)[3 + \alpha_1 - \pi_1(1-\alpha_1)/(1-\pi_1)] .$$

If $\alpha_1 \neq 1$

$$3(1-\alpha_1)[1+\pi_1/(1-\pi_1)] = 3 + \alpha_1 - \pi_1(1-\alpha_1)/(1-\pi_1),$$

$$4\pi_1/(1-\pi_1) = 4\alpha_1/(1-\pi_1) .$$

Therefore $\alpha_1 = \pi_1$.

From (2.51)

$$\alpha_2 = 1 - \pi_1(1-\pi_1)/\pi_2 .$$

Therefore $\alpha_2 = \pi_2$.

This completes the proof since we have already shown the condition $\alpha_1 = \pi_1$, $\alpha_2 = \pi_2$ is sufficient to yield the uniform distribution.

We now consider the distributions for $\alpha_1 + \alpha_2 > 1$. Under this restriction on the alphas, the functional system (2.26) - (2.28) no longer lends itself to a unique construction of the limiting distribution functions. However, the following theorem shows that the functions satisfying the functional system are unique distribution functions.

<u>Theorem 2.3</u> The functional system (2.26) - (2.28) uniquely defines the distribution function corresponding to the characteristic functions which satisfy equation (2.19).

<u>Proof:</u> Lemma 2.3 shows that the moments generated by the characteristic functional equation (2.19) are unique so the characteristic functions satisfying (2.19) are also unique. Furthermore, the inversion of the characteristic functional equation yields the unique functional equation (2.26).

Suppose there are two functions $F(x)$ and $G(x)$, both of which satisfy the functional system. Then the moments corresponding to these two functions must be the moments defined by (2.19). Since $F(x)$ and

$G(x)$ have identical moments, $F(x) \equiv G(x)$ by the uniqueness theorem of characteristic functions. ([11] p.35).

<u>Theorem 2.4</u> For $\alpha_1 + \alpha_2 > 1$ there exists an absolutely continuous function $F(x)$ which satisfies the functional equation

$$F(x) = \pi_1 F((x+1-\alpha_1)/\alpha_1) + \pi_2 F((x-1+\alpha_2)/\alpha_2)$$

for all x in $(-\infty, 2\alpha_1 - 1)$ and also satisfies the boundary conditions

$$F(x) = 0, \quad x \leq -1 \; ; \quad F(x) = 1 \; .$$

This function is piecewise in the sense that the functional form is different over each interval of a set of intervals $(b_m, b_{m+1}) = I_m$ $(m=1,2,3,\ldots,M)$ covering $(-1, 1)$. In the interval I_m, the function takes the form

$$F_m(x) = c[(x+1)^p + \sum_{i=2}^{m} K_i (x-b_i)^p] , \tag{2.52}$$

where $p = \ln \pi_1 / \ln \alpha_1$, $\{K_i\}$ are constants and c is a normalizing constant.

<u>Proof:</u> In the open interval $(-1, 1)$ the arguments of the functional equation (2.26) are ordered such that

$$(x+1-\alpha_1)/\alpha_1 > x > (x-1+\alpha_2)/\alpha_2 \; .$$

Making the change of variable $u = (x+1-\alpha_1)/\alpha_1$ transforms the functional equation to

$$F(\alpha_1 u - 1 + \alpha_1) = \pi_1 F(u) + \pi_2 F((\alpha_1 u + \alpha_1 + \alpha_2 - 2)/\alpha_2) \; .$$

Re-arranging the terms in order of decreasing argument

$$\pi_1 F(u) = F(\alpha_1 u - 1 + \alpha_1) - \pi_2 F((\alpha_1 u + \alpha_1 + \alpha_2 - 2)/\alpha_2) \ . \tag{2.53}$$

As u increases from -1, there exists an interval such that $(\alpha_1 u + \alpha_1 + \alpha_2 - 2)/\alpha_2 \leq -1$. This interval is $(-1 \leq u \leq (2 - 2\alpha_2 - \alpha_1)/\alpha_1)$, and we shall denote it by I_1. For $u \in I_1$, the lower boundary condition reduces (2.53) to

$$\pi_1 F(u) = F(\alpha_1 u - 1 + \alpha_1) \ , \qquad (u \in I_1) \ . \tag{2.54}$$

In order to meet the condition $F(-1) = 0$, we try a solution of the form

$$F_1(u) = c(u+1)^p \ , \qquad (u \in I_1) \ . \tag{2.55}$$

Substituting (2.55) into (2.54), the left member gives

$$c\pi_1 (u+1)^p$$

and the right member yields

$$c(\alpha_1 u + \alpha_1)^p \ .$$

Rewriting the above term in the form

$$c\alpha_1{}^p (u + 1)^p$$

it is obvious the trial function will satisfy (2.54) if

$$\alpha_1{}^p = \pi_1 \ .$$

Thus we must have

(2.56) $$p = \ln \pi_1/\ln \alpha_1 \quad .$$

For u greater than the upper boundary $(2-2\alpha_2-\alpha_1)/\alpha_1$ of I_1, there exists a second interval I_2 such that for $u \in I_2$, both the arguments $(\alpha_1 u-1+\alpha_1)$ and $(\alpha_1 u+\alpha_1+\alpha_2-2)/\alpha_2$ lie in I_1. The upper boundary of the interval I_2 is determined by the condition $\alpha_1 u-1+\alpha_1 = (2-2\alpha_2-\alpha_1)/\alpha_1$. Thus $I_2 = [(2-2\alpha_1-\alpha_1)/\alpha_1 \, , \, (2-2\alpha_2-\alpha_1^2)/\alpha_1^2]$.

The function $F_2(u)$ $(u \in I_2)$ can be constructed from (2.53) by substituting the known function (2.55) into the right hand member of the functional equation (2.53). Following this procedure we have

$$\pi_1 F_2(u) = c\{\alpha_1^p(u+1)^p - \pi_2[(\alpha_1 u-\alpha_1+\alpha_2-2)/\alpha_2+1]^p\} \, ,$$

$$= c\{\alpha_1^p(u+1)^p - \pi_2(\alpha_1/\alpha_2)^p[u+(\alpha_1+\alpha_2-2)/\alpha_1+\alpha_2/\alpha_1]^p\} \, .$$

Since $\alpha_1^p = \pi_1$

(2.57) $$F_2(u) \;= c\{(u+1)^p - \pi_2/\alpha_2^p\,[u-(2-2\alpha_1-\alpha_1)/\alpha_1]^p\} \, , \quad (u \in I_2) \, .$$

Let $b_2 = (2 - 2\alpha_2 - \alpha_1)/\alpha_1$ and $K_2 = -\pi_2/\alpha_2^p$.

Thus (2.57) is of the form

$$F_2(u) \;= c\{(u+1)^p + K_2(u-b_2)^p\} \, ,$$

where b_2 is the lower bound of I_2 (c.f. above).

In the third interval I_3 , the situation becomes slightly more complicated. The upper boundary of I_3 is determined by either

i) the argument $\alpha_1 u - 1 + \alpha_1$ crossing into I_3

ii) the argument $(\alpha_1 u + \alpha_1 + \alpha_2 -2)/\alpha_1$ crossing into I_2 ,

whichever occurs at the smallest value of u . Thus the upper boundary of I_3 is given by

$$\mathrm{Min}[(2-2\alpha_2-\alpha_1^3)/\alpha_1^3 \quad , \; 2(\alpha_1+\alpha_2-\alpha_1^2-\alpha_2^2-\alpha_1\alpha_2)/\alpha_1^2] \quad .$$

The function $F_3(u)$ $(u \in I_3)$ is of the form

$$F_3(u) = c\{(u+1)^p+K_2(u-b_2)^p+K_3(u-b_3)^p\}$$

where $b_3 = (2 - 2\alpha_2 - \alpha_1^2)/\alpha_1^2$.

This may be verified by substituting into (2.53) using the fact that the argument $(\alpha_1 u-1+\alpha_1) \in I_2$ and the argument $(\alpha_1 u+\alpha_1+\alpha_2-2)/\alpha_2 \in I_1$.

The calculation of $F_4(u)$ proceeds routinely as above unless the two smaller arguments both cross boundary points simultaneously to give the same value of the upper boundary b_4 of I_3 ; that is, if

$$(2-2\alpha_2-\alpha_1^3)/\alpha_1^3 \quad = \quad 2(\alpha_1+\alpha_2-\alpha_1^2-\alpha_2^2-\alpha_1\alpha_2)/\alpha_1^2 \quad .$$

If this possibility occurs, then the point so determined may or may not be a true boundary point. Calculating F_4 gives

$$F_4(u) = c\{(u+1)^p + K_2(u-(b_2-1+\alpha_1)/\alpha_1)^p + K_3(u-(b_3+1-\alpha_1)/\alpha_1)^p$$
$$- \pi_2/\alpha_2^p[(u-b_2)^p + K_2(u-(\alpha_2 b_2+2-\alpha_1-\alpha_2)/\alpha_1)^p]\} \quad ,$$

where

$$(u-(b_3+1-\alpha_1)/\alpha_1)^p = (u-(\alpha_2 b_2+2-\alpha_1-\alpha_2)/\alpha_1)^p \quad .$$

If $F_4(u)$ does not reduce to $F_3(u)$, that is if $K_3 \neq (\pi_2/\alpha_2{}^p)K_2$, then the point determined is a true boundary and the two terms $K_3(u-(b_3+1-\alpha_1)/\alpha_1)^p$ - $(\pi_2/\alpha_2{}^p)K_2(u-(\alpha_2 b_2+2-\alpha_1-\alpha_2)/\alpha_1)^p$ may be collected into one term and written in the form $K_4(u-b_4)^p$. However, if $K_3 = (\pi_2/\alpha_2{}^p)K_2$, then the two terms will cancel, in which case we call the point generated a pseudo-boundary point, and the true value of b_4 is given by the value of u specified when the smallest argument crosses b_3 .

Now we wish to find the function $F_m(u)$ over the interval $I_m=(b_m, b_{m+1})$. Suppose $\alpha_1 u-1+\alpha_1 \in I_{i-1}$ and $(\alpha_1 u+\alpha_1+\alpha_2-2)/\alpha_2 \in I_{j-1} (i \geq j)$, while $u \in I_m$. The upper boundary of I_m is given by

$$b_{m+1} = \text{Min}[b_i+1-\alpha_1)/\alpha_1, \ (\alpha_2 b_j+2-\alpha_1-\alpha_2)/\alpha_1] \ . \tag{2.58}$$

If the point determined by (2.58) is a pseudo-boundary point, the true boundary point will be given by

$$b_{m+1} = \text{Min}[(b_{i+1}+1-\alpha_1)/\alpha_1 \ , \ (\alpha_2 b_{j+1}+2-\alpha_1-\alpha_2)/\alpha_1].$$

Since m intervals have been generated at this stage, the arguments $(\alpha_1 u-1+\alpha_1)$ and $(\alpha_1 u+\alpha_1+\alpha_2-2)/\alpha_2$ must together have crossed $m+\nu+2\mu$ boundaries; where ν is the number of simultaneous crossing of two arguments which give true boundary points and μ is the number of pseudo-boundary points generated. Thus

$$m = i + j - 2 - \nu - 2\mu \ .$$

An induction argument may now be used to prove that $F_m(u)(u \in I_m)$ has the form

$$F_m(u) = c\{(u+1)^p + \sum_{k=2}^{m} K_k(u-b_k)^k\} , \qquad (b_m \leq u < b_{m+1}) .$$

The induction hypothesis has already been proved for $m = 1$ and 2.
Assume that the hypothesis is true for all the functions $F_k(u)$ $(k=1,2,\ldots,m-1)$.
In particular

$$F_{j-1}(u) = c\{(u+1)^p + \sum_{k=2}^{j-1} K_k(u-b_k)^p\}, \qquad (u \in I_{j-1}) ;$$

$$F_{i-1}(u) = c\{(u+1)^p + \sum_{k=2}^{i-1} K_k(u-b_k)^p\}, \qquad (u \in I_{i-1}) .$$

Substituting into (2.53) gives

$$(2.59) \qquad F_m(u) = c\{(u+1)^p + \sum_{k=2}^{i-1} K_k(u-(b_k+1-\alpha_1)/\alpha_1)^p$$

$$- \pi_2/\alpha_2{}^p[(u-b_2)^p + \sum_{k=2}^{j-1} K_k(u-(\alpha_2 b_k+2-\alpha_1-\alpha_2)/\alpha_1)^p]\} , \quad (u \in I_m) .$$

The $(j-2)$ values $\{(b_k+1-\alpha_1)/\alpha_1\}$ are the endpoints generated when $\alpha_1 u-1+\alpha_1$ crosses into the intervals $\{I_k\}$ (cf. (2.58)). Similarly, the terms $\{(\alpha_1 b_k+2-\alpha_1-\alpha_2)/\alpha_1\}$ are the endpoints which are generated when $(\alpha_1 u+\alpha_1+\alpha_2-2)/\alpha_2$ crosses into the intervals $\{I_k\}$. There are only $i+j-2-\nu-2\mu$ independent values, since 2ν terms will collapse into ν terms and 2μ terms will vanish. Thus the right hand side of (2.59) can

be written in the form

$$F_m(u) = c\{(u+1)^p + \sum_{k=2}^{m} K_k(u-b_k)^p\} \quad , \quad (u \in I_m) \; ;$$

and the induction is complete.

To finish specifying the function $F(x)$, two more details are required:

i) Let $F_o(x) = 0$, $x \in (-\infty, -1)$,

ii) In the final interval $I_m = (b_m, 1)$, evaluate c by the condition $F_m(1) = 1$.

This completes theorem 2.4.

The intervals generated by the above procedure may be illustrated geometrically as follows.

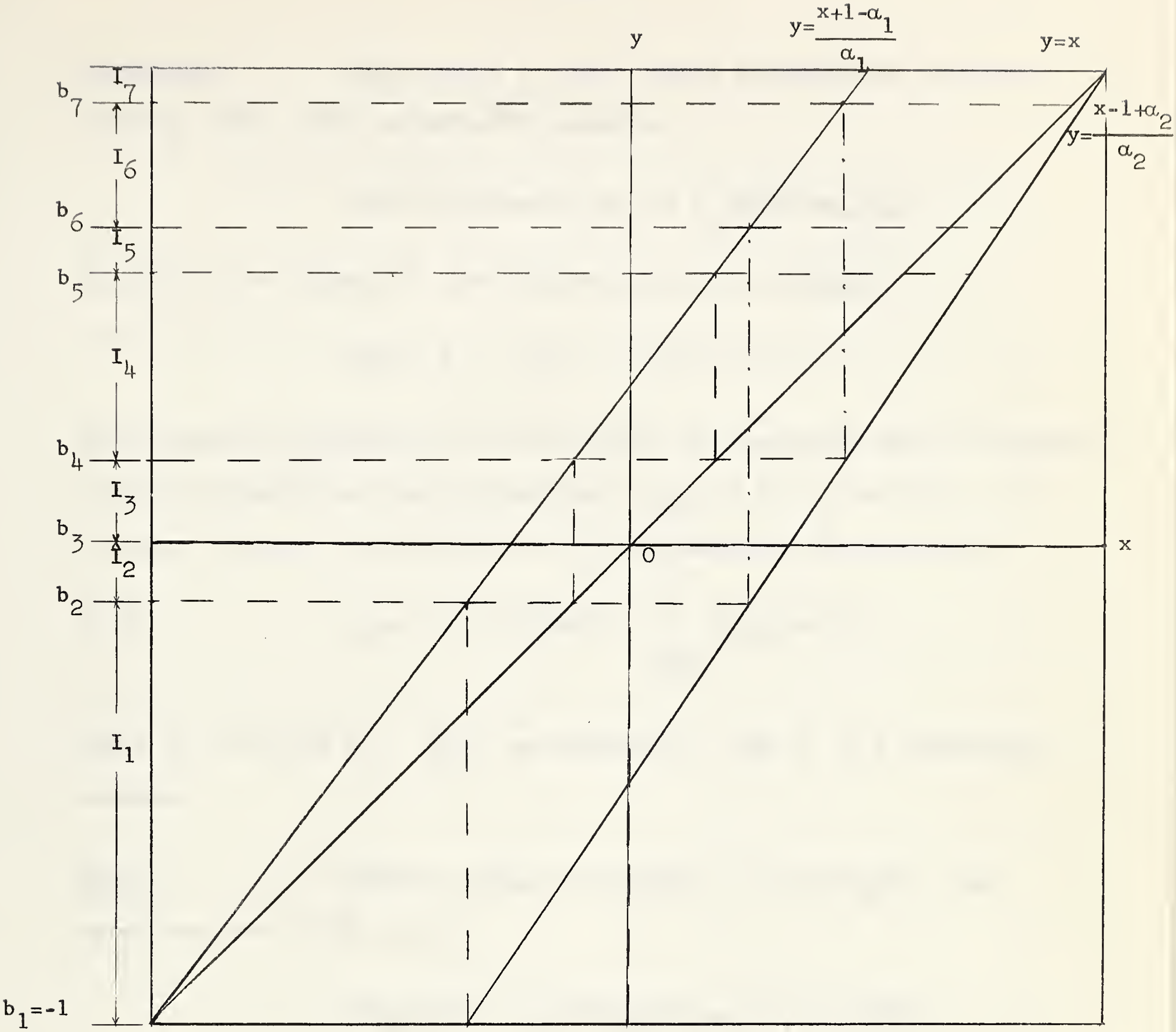

Figure 3: Geometric Interpretation of the Method of Generating the intervals I_i .

The upper boundary of the m^{th} interval is given by the value of the argument $(x+1-\alpha_1)/\alpha_1$ when either x or $(x-1+\alpha_2)/\alpha_2$ crosses a a boundary point already determined. For example, the upper boundary of I_1 is given by the value of $(x+1-\alpha_1)/\alpha_1$ when the argument $(x-1+\alpha_2)/\alpha_2$ intersects the boundary $b_1 = -1$. Similarly, the upper boundary of I_2 is defined when x intersects b_2 . The procedure for finding the boundary points continues along in the fashion illustrated until the upper boundary of I_m is greater than or equal to $+1$.

Theorem 2.5 For $\alpha_1+\alpha_2 > 1$ there exists an absolutely continuous function $G(x)$ which satisfies the equation

$$G(x) = \pi_1 G((x+1-\alpha_1)/\alpha_1) + \pi_2\, G((x-1+\alpha_2)/\alpha_2)$$

for all x in $(1-2\alpha_2, \infty)$ and also satisfies the conditions

$$G(x) = 1 \quad , \quad x \geq 1 \quad ; \quad G(-1) = 0 \ .$$

This function is piecewise in the sense that the functional form is different over each interval of a set of intervals $(d_{n+1}, d_n) = J_n$ $(n=1,2,3,\ldots N)$ covering $(-1, 1)$. In the interval J_n the function takes the form

$$G_n(x) = 1 - B[(1-x)^q + \sum_{i=2}^{n} K_i(d_i-x)^q] \quad , \tag{2.60}$$

where $q = \ln \pi_2/\ln \alpha_2$, $\{K_i\}$ are constants, and B is a normalizing constant.

Proof: Make the change of variable $\nu = (x-1+\alpha_2)/\alpha_2$ and substitute into (2.26) to get

$$G(\alpha_2\nu+1-\alpha_2) = \pi_1 G((\alpha_2\nu-2-\alpha_1-\alpha_2)/\alpha_1) + \pi_2 F(\nu) \ .$$

Rewriting in order of increasing arguments

$$\pi_2 G(\nu) = G(\alpha_2\nu+1-\alpha_2) - \pi_1\, G((\alpha_2\nu+2-\alpha_1-\alpha_2)/\alpha_1) \ . \tag{2.61}$$

In the interval $[-1, 1]$ there exists an interval such that $(\alpha_2\nu+2-\alpha_1-\alpha_2)/\alpha_1 \geq 1$. Denote this interval $[(2\alpha_1+\alpha_2-2)/\alpha_2, 1]$ by J_1 . Imposing the condition $G(\nu) = 1$, $\nu \geq 1$ reduces (2.61) to

$$\pi_2 G(\nu) = G(\alpha_2\nu+1-\alpha_2) - \pi_1 \;, \qquad (\nu \in J_1)\;. \tag{2.62}$$

To meet the condition $G(1) = 1$, try a solution of the form

$$G_1(\nu) = 1 - B(1-\nu)^q \;, \qquad (\nu \in J_1)\;. \tag{2.63}$$

Substituting the above function into (2.62), the left member is

$$\pi_2 - \pi_2 B(1-\nu)^q \;,$$

and the right member becomes

$$1 - B(\alpha_2-\alpha_2\nu)^q - \pi_1 \;.$$

The above term can be written

$$\pi_2 - B\alpha_2{}^q(1-\nu)^q \;.$$

Thus the trial function will satisfy (2.62) if

$$\alpha_2{}^q = \pi_2 \;.$$

This imposes the condition

$$q = \ln \pi_2/\ln \alpha_2 \;. \tag{2.64}$$

For ν less than the lower bound of $J_1 = (2\alpha_1+\alpha_2-2)/\alpha_2$, there exists a second interval J_2 such that for $\nu \in J_2$, both the arguments $(\alpha_2\nu+1-\alpha_2)$ and $(\alpha_2\nu+2-\alpha_1-\alpha_2)/\alpha_1$ are in J_1. The lower boundary of J_2 is determined by the condition

$$\alpha_2\nu+1-\alpha_2 = (2\alpha_1+\alpha_2-2)/\alpha_2 \;.$$

Thus J_2 is the interval $[(2\alpha_1+\alpha_2{}^2-2)/\alpha_2{}^2 \;,\; 2\alpha_1+\alpha_2-2]$. The function

G_2 defined on the interval J_2 may be found by substituting G_1 into the right hand side of (2.61) to get

$$\pi_2 G_2(\nu) = 1-B\alpha_2{}^q(1-\nu)^q-\pi_1[1-B(\alpha_2/\alpha_1)^q((2\alpha_1+\alpha_2-2)/\alpha_2-\nu)^q] .$$

Therefore

$$G_2(\nu) = 1-B[(1-\nu)^q-(\pi_1/\alpha_1{}^q)((2\alpha_1+\alpha_2-2)/\alpha_2-\nu)^q] .$$

Denote the upper boundary of J_2 by d_2 and let $K_2 = -\pi_1/\alpha_1{}^q$. With these substitutions, the functional form of $G_2(\nu)$ becomes

$$G_2(\nu) = 1-B[(1-\nu)^q+K_2(d_2-\nu)^q] , \qquad (\nu \in J_2) .$$

In the general case, suppose that the argument $(\alpha_2\nu+1-\alpha_2) \in J_{i-1}$ and the argument $((\alpha_2\nu+2-\alpha_1-\alpha_2)/\alpha_1) \in J_{j-1}$ $(i \geq j)$ while $\nu \in J_n$. The lower bound of J_n is given by

$$d_n = \mathrm{Min}[(d_i-1+\alpha_2)/\alpha_2 \;,\; (\alpha_1 d_j+\alpha_1+\alpha_2-2)/\alpha_2] . \tag{2.65}$$

However, if the value of d_n determined by (2.65) turns out to be a pseudo-boundary point, then the true boundary point d_n will be

$$d_n = \mathrm{Min}[(d_{i+1}-1+\alpha_2)/\alpha_2, \; (\alpha_1 d_{j+1}+\alpha_1\alpha_2-2)/\alpha_2] .$$

The same induction argument used in theorem 2.4 may be applied to prove the form of G_n is

$$G_n(\nu) = 1-B[(1-\nu)^q + \sum_{i=2}^{n} K_i(d_i - \nu)^q] , \quad (\nu \in J_n) .$$

$G(x)$ is completely specified by the additional conditions

$$G_o(x) = 1 \quad , \quad (1 \leq x < \infty)$$

and by evaluating B by the boundary value

$$G_N(-1) = 0 \ ,$$

where J_N is the interval which contains the point -1.

<u>Theorem 2.6</u> If the functions $F(x)$ and $G(x)$ of the previous two theorems are identical over $(-1, 1)$, then we have constructed the unique absolutely continuous distribution function which satisfies the functional system (2.26)-(2.28).

<u>Proof:</u> That the functional equation is satisfied everywhere follows from the construction. The proof then follows immediately from theorem 2.3 which shows that the functional system (2.26) - (2.28) defines the unique distribution function corresponding to the unique characteristic function which satisfies (2.19).

If the construction of theorems 2.4 and 2.5 fails for a given set of alphas and pis, then the distribution function may have some other absolutely continuous form than the one constructed, or it may be a purely singular function.

CHAPTER III

THE SUBJECT-CONTROLLED MODEL

§ 3.1 Introduction

This chapter contains a treatment of the Subject-Controlled Model which follows much the same outline as Chapter II. First the basic model is derived, then the equation satisfied by the characteristic functions of the limiting distributions is found. We again standardize the characteristic equation by mapping the limiting distribution on the domain $(-1,1)$. However, in this case the standardizing procedure gives two transformed characteristic equations depending upon the relative sizes of λ_1 and λ_2. Applying the fundamental inversion theorem we find that the transformed limiting distributions satisfy one of two functional integral equations. A study of these functional integral equations reveals a few of the properties of the limiting distributions.

§ 3.2 Derivation of the Model

The Subject-Controlled Events Model will be derived in a similar manner to the Experimenter Model of Chapter II. For this model, also, we shall consider a simple learning experiment in which each trial consists of presenting a subject with two mutually exclusive alternatives A_1 and A_2. As before, the choice of A_1 is called response R_1 and the choice of A_2 is called R_2. The major difference is that outcome O_1 is always associated with A_1 and outcome O_2 is always associated with A_2. Under this scheme the mutually exclusive events are

$$
(3.1) \qquad \begin{aligned} E_1 &: R_1 \text{ followed by } O_1 , \\ E_2 &: R_2 \text{ followed by } O_2 . \end{aligned}
$$

In this type of experiment it is obvious that the outcome of any trial is wholly determined by the response of the subject.

Following the notation introduced in Chapter II, let p_n be the probability of response R_1 and $q_n = 1 - p_n$ be the probability of response R_2 at the n^{th} trial. The response probabilities p_n and q_n are transformed by the same operators Q_1 and Q_2 that were used in the Experimenter case, that is

$$
(3.2) \qquad p_{n+1} = Q_1 p_n = a_1 + \alpha_1 p_n ,
$$

if E_1 occurred at trial n ; or

$$
(3.3) \qquad p_{n+1} = Q_2 p_n = a_2 + \alpha_2 p_n ,
$$

if E_2 occurred at trial n. The parameters $a_1, a_2, \alpha_1, \alpha_2$ satisfy the same conditions as before. (c.f. 2.7)

Since the subject determines which event occurs at the n^{th} stage, the probability of applying the operators Q_1 and Q_2 is no longer a constant π_1 or π_2 . Now it becomes a function of p_n such that the conditional probability of E_1 occurring on the $n+1^{st}$ trial given that E_1 occurred on the n^{th} trial is $Q_1 p_n$, and the conditional probability of E_1 occurring on the $n+1^{st}$ trial given E_2 on the n^{th} is $Q_2 p_n$.

As in the Experimenter Case, if the subject has an initial response probability p_o, the possible values of p_n throughout the first few trials

are generated as shown in figure 4.

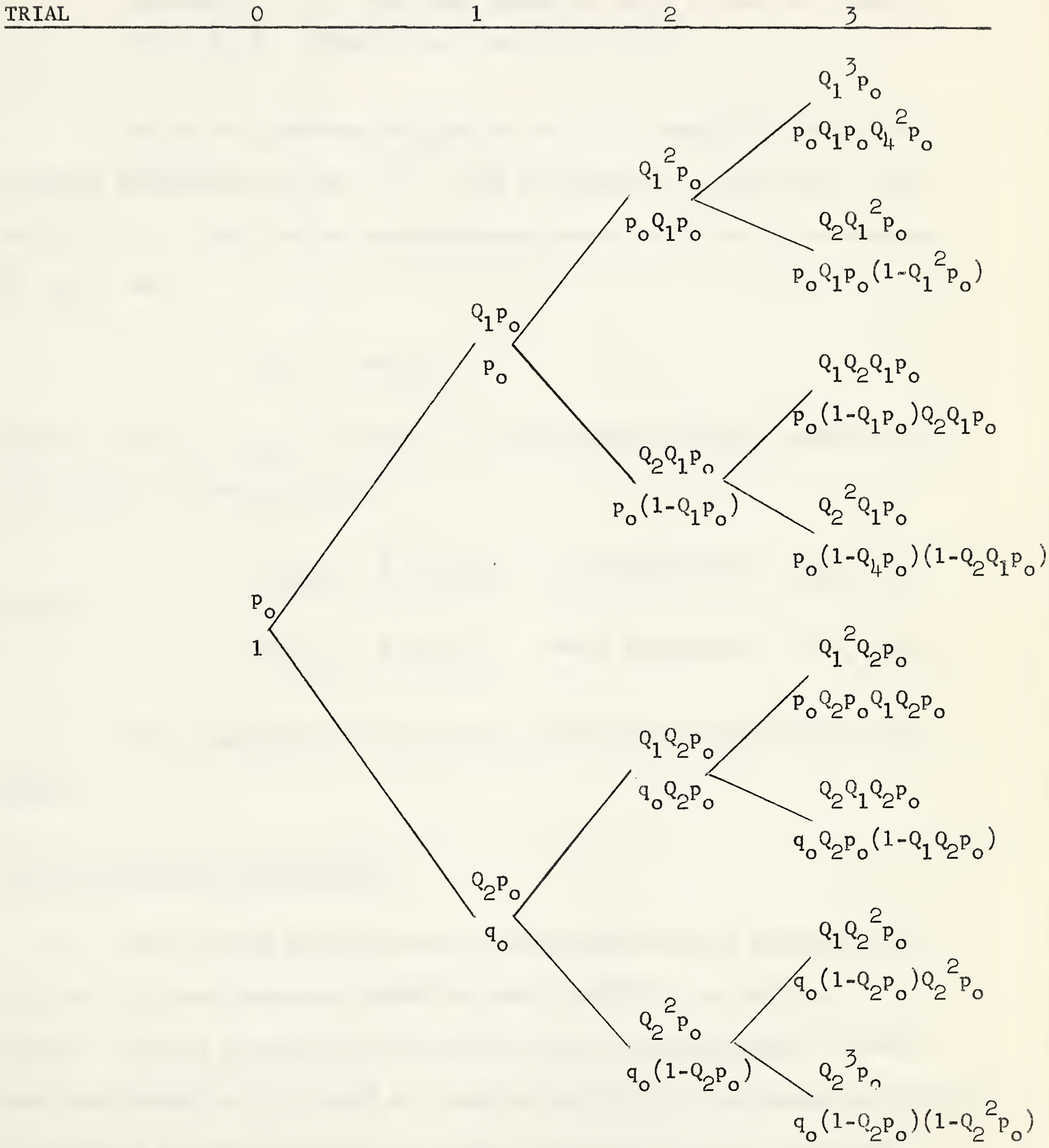

Figure 4: Possible values of the response probability p_n and the probability of the occurrence of that particular value of p_n throughout the

first three trials. The upper entry is the value of p_n in terms of the operators Q_1 and Q_2 and the initial response probability p_o. The lower entry is $Pr(p_n)$ also in terms of Q_1, Q_2, p_o, q_o. (Reproduced from [1] p. 79)

As in the previous chapter, let the 2^n possible values of the response probability at the n^{th} trial be denoted $p_{m,n}(m=1,2,3,\ldots,2^n; n=1,2,3,\ldots)$. Also let the corresponding probabilities of the occurrence of $p_{m,n}$ be

$$P_{m,n} = Pr(p_{m,n}) \ .$$

Thus if $Pr(R_1) = p_{m,n}$ on trial n, the possible response probabilities for trial $n+1$ are given by

$$(3.4) \qquad \begin{aligned} Q_1 p_{m,n} &= a_1 + \alpha_1 p_{m,n} && \text{with probability } p_{m,n} P_{m,n}, \\ Q_2 p_{m,n} &= a_2 + \alpha_2 p_{m,n} && \text{with probability } (1-p_{m,n}) P_{m,n} \ . \end{aligned}$$

This completes the derivation of the basic Subject-Controlled Model.

§ 3.3 Properties of the Model

The varying probabilities of the application of the operators Q_1 and Q_2 have made this model extremely difficult to analyze. As a result, very few properties of this model have been determined. However Bush and Mosteller [12] found an equation satisfied by the moment generating function of the distribution of response probabilities by an approach

similar to that of the Experimenter Case.

Let $m_n(\theta)$ be the moment generating function of the distribution of $p_{m,n}$. By definition

$$m_{n+1}(\theta) = \sum_{m=1}^{2^{n+1}} e^{\theta p_{m,n+1}} P_{m,n+1} ,$$

$$= \sum_{m=1}^{2^n} \left[e^{(a_1+\alpha_1 p_{m,n})\theta} p_{m,n} P_{m,n} + e^{(a_2+\alpha_2 p_{m,n})\theta} (1-p_{m,n}) P_{m,n} \right] ,$$

$$= e^{a_1\theta} \sum_{m=1}^{2^n} \left[e^{\alpha_1 p_{m,n}\theta} p_{m,n} P_{m,n} \right] + e^{a_2\theta} \sum_{m=1}^{2^n} \left[e^{\alpha_1 p_{m,n}\theta} P_{m,n} \right]$$

$$\cdot e^{a_2\theta} \sum_{m=1}^{2^n} \left[e^{\alpha_2 p_{m,n}\theta} p_{m,n} P_{m,n} \right] .$$

Therefore

$$m_{n+1}(\theta) = \left(\frac{1}{\alpha_1}\right) e^{a_1\theta} \frac{d}{d\theta}[m_n(\alpha_1\theta)] + e^{a_2\theta} m_n(\alpha_2\theta) - \left(\frac{1}{\alpha_2}\right) e^{a_2\theta} \frac{d}{d\theta}[m_n(\alpha_2\theta)] . \tag{3.5}$$

Thus in the Subject Case, the moment generating function of the distribution of response probabilities at the $n+1^{st}$ trial is dependent not only on all the moments of all the previous trials but also upon the first derivatives of all the previous moment generating functions.

Replacing a_i by $\lambda_i(1-\alpha_i)$ $(i=1,2)$, equation (3.5) becomes

$$(3.6) \qquad m_{n+1}(\theta) = \left(\frac{1}{\alpha_1}\right)e^{\lambda_1(1-\alpha_1)\theta}\frac{d}{d\theta}[m_n(\alpha_1\theta)]+e^{\lambda_2(1-\alpha_2)\theta}m_n(\alpha_2\theta)$$

$$-\left(\frac{1}{\alpha_2}\right)e^{\lambda_2(1-\alpha_2)\theta}\frac{d}{d\theta}[m_n(\alpha_2\theta)] .$$

Converting the moment-generating functional equation (3.6) to a characteristic functional equation gives

$$(3.7) \qquad \varphi_{n+1}(\theta) = (-i/\alpha_1)e^{i\lambda_1(1-\alpha_1)\theta}\frac{d}{d\theta}[\varphi_n(\alpha_1\theta)]+e^{i\lambda_2(1-\alpha_2\,\theta}\varphi_n(\alpha_2\theta)$$

$$+ (i/\alpha_2)e^{i\lambda_2(1-\alpha_2)\theta}\frac{d}{d\theta}[\varphi_n(\alpha_2\theta)] ,$$

where $\varphi_n(\theta) = m_n(i\theta)$.

Harris [1 p.99] has shown that a limiting distribution $F(p)$ exists as $n \to \infty$ except in the cases where either or both of $\lambda_2 = 0$ and $\lambda_1 = 1$ hold. In these exceptional cases, all of the probability density will be concentrated at $p = 0$ or $p = 1$.

To begin the investigation of the limiting distributions, let

$$\lim_{n\to\infty} \varphi_n(\theta) = \varphi(\theta)$$

The characteristic function of the limiting distribution of response probabilities satisfies the functional differential equation

$$(3.8) \qquad \varphi(\theta) = (-i/\alpha_1)e^{i\lambda_1(1-\alpha_1)\theta}\frac{d}{d\theta}[\varphi(\alpha_1\theta)]+e^{i\lambda_2(1-\alpha_2)\theta}\varphi(\alpha_2\theta)+(i/\alpha_2)$$

$$\times e^{i\lambda_2(1-\alpha_2)\theta}\frac{d}{d\theta}[\varphi(\alpha_2\theta)] .$$

As Karlin's trapping theorem also applies to this problem, the boundary conditions on $F(p)$ are

$$(3.9)\qquad \begin{aligned} F(p) &= 0 \quad , \quad p \leq \mathrm{Min}(\lambda_1,\lambda_2) \ , \\ F(p) &= 1 \quad , \quad p \geq \mathrm{Max}(\lambda_1,\lambda_2) \ . \end{aligned}$$

Following the procedure of the Experimenter Model we will make a transformation to map the limiting distribution onto the interval [-1,1]. However (3.8) is not symmetric in α_1,α_2 and λ_1,λ_2 so the following transformations must be applied.

$$(3.10)\qquad x = (\lambda_1+\lambda_2-2p)/(\lambda_1-\lambda_2) \ , \qquad \text{for } \lambda_2 > \lambda_1 \ ,$$

$$(3.11)\qquad x = (\lambda_2+\lambda_1-2p)/(\lambda_2-\lambda_1) \ , \quad \text{for } \lambda_1 > \lambda_2, \ (\lambda_1 \neq 1, \lambda_2 \neq 0) \ .$$

Equation (2.18) gives

$$\varphi_p(\alpha\theta) = e^{\frac{1}{2}i\alpha\theta(\lambda_2+\lambda_1)} \varphi_x[\tfrac{1}{2}\alpha\theta(\lambda_2-\lambda_1)] \ , \qquad (\lambda_2 > \lambda_1) \ .$$

Thus for $\lambda_2 > \lambda_1$

$$\frac{d}{d\theta}[\varphi_p(\alpha\theta)] = \frac{d}{d\theta}\{e^{\frac{1}{2}i\alpha\theta(\lambda_2+\lambda_1)} \varphi_x[\tfrac{1}{2}\alpha\theta(\lambda_2-\lambda_1)]\} \ ,$$

or

$$(3.12)\qquad \begin{aligned} \frac{d}{d\theta}[\varphi_p(\alpha\theta)] &= \tfrac{1}{2}i\alpha(\lambda_2+\lambda_1)e^{\frac{1}{2}i\alpha\theta(\lambda_2+\lambda_1)} \varphi_x[\tfrac{1}{2}\alpha\theta(\lambda_2-\lambda_1)] \\ &\quad + e^{\frac{1}{2}\alpha\theta(\lambda_2+\lambda_1)} \frac{d}{d\theta}\{\varphi_x[\tfrac{1}{2}\alpha\theta(\lambda_2-\lambda_1)]\} \ . \end{aligned}$$

The characteristic equation corresponding to transformation (3.10) may be derived by substituting (2.18) and (3.12) into (3.8) to give

$$e^{\frac{1}{2}\theta(\lambda_2+\lambda_1)}\varphi_x[\tfrac{1}{2}\theta(\lambda_2-\lambda_1)]=\tfrac{1}{2}(\lambda_2+\lambda_1)e^{i\theta[\lambda_1(1-\alpha_1)+\frac{1}{2}\alpha_1(\lambda_2+\lambda_1)]}\varphi_x[\tfrac{1}{2}\alpha_1\theta(\lambda_2-\lambda_1)]$$

$$-(i/\alpha_1)e^{i\theta[\lambda_1(1-\alpha_1)+\frac{1}{2}\alpha_1(\lambda_2+\lambda_1)]}\frac{d}{d\theta}\{\varphi_x[\tfrac{1}{2}\alpha_1\theta(\lambda_2-\lambda_1)]\}$$

$$+e^{i\theta[\lambda_2(1-\alpha_2)+\frac{1}{2}\alpha_2(\lambda_2+\lambda_1)]}\varphi_x[\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)]$$

$$-\tfrac{1}{2}(\lambda_2+\lambda_1)e^{i\theta[\lambda_2(1-\alpha_2)+\frac{1}{2}\alpha_2(\lambda_2+\lambda_1)]}\varphi_x[\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)]$$

$$+(i/\alpha_2)e^{i\theta[\lambda_2(1-\alpha_2)+\frac{1}{2}\alpha_2(\lambda_2+\lambda_1)]}\frac{d}{d\theta}\{\varphi_x[\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)]\}\ .$$

Thus

$$\varphi_x[\tfrac{1}{2}\theta(\lambda_2-\lambda_1)]=\tfrac{1}{2}(\lambda_2+\lambda_1)e^{\frac{1}{2}i\theta(1-\alpha_1)(\lambda_1-\lambda_2)}\varphi_x[\tfrac{1}{2}\alpha_1\theta(\lambda_2-\lambda_1)]$$

$$-(i/\alpha_1)e^{\frac{1}{2}i\theta(1-\alpha_1)(\lambda_1-\lambda_2)}\frac{d}{d\theta}\{\varphi_x[\tfrac{1}{2}\alpha_1\theta(\lambda_2-\lambda_1)]\}$$

$$+[1-\tfrac{1}{2}(\lambda_2+\lambda_1)]\,e^{\frac{1}{2}i\theta(1-\alpha_2)(\lambda_2-\lambda_1)}\varphi_x[\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)]$$

$$+(i/\alpha_2)e^{\frac{1}{2}i\theta(1-\alpha_2)(\lambda_2-\lambda_1)}\frac{d}{d\theta}\{\varphi_x[\tfrac{1}{2}\alpha_2\theta(\lambda_2-\lambda_1)]\}\ .$$

Writing $u=\frac{1}{2}\theta(\lambda_2-\lambda_1)$ and omitting the subscript x on φ we have

$$\varphi(u)=\tfrac{1}{2}(\lambda_2+\lambda_1)e^{-iu(1-\alpha_1)}\varphi(\alpha_1u)+[1-\tfrac{1}{2}(\lambda_2+\lambda_1)]e^{iu(1-\alpha_2)}\varphi(\alpha_2u)$$
$$-[\tfrac{1}{2}i(\lambda_2-\lambda_1)/\alpha_1]e^{-iu(1-\alpha_1)}\frac{d}{du}[\varphi(\alpha_1u)]$$
$$+[\tfrac{1}{2}i(\lambda_2-\lambda_1)/\alpha_2]e^{iu(1-\alpha_2)}\frac{d}{du}[\varphi(\alpha_2u)]\ ,\quad(\lambda_2>\lambda_1)\ . \tag{3.13}$$

The characteristic equation corresponding to transformation (3.11) is similarly derived by interchanging λ_1 and λ_2 in (2.18) and (3.12) then substituting into (3.8) to get

$$e^{\frac{1}{2}i\theta(\lambda_1+\lambda_2)}\varphi[\tfrac{1}{2}\theta(\lambda_1-\lambda_2)] = \tfrac{1}{2}(\lambda_1+\lambda_2)e^{i\theta[\lambda_1(1-\alpha_1)+\frac{1}{2}\alpha_1(\lambda_1+\lambda_2)]}\varphi[\tfrac{1}{2}\alpha_1\theta(\lambda_1-\lambda_2)]$$

$$-(i/\alpha_1)e^{i\theta[\lambda_1(1-\alpha_1)+\frac{1}{2}\alpha_1(\lambda_1+\lambda_2)]}\frac{d}{d\theta}\{\varphi[\tfrac{1}{2}\alpha_1\theta(\lambda_1-\lambda_2)]\}$$

$$-\tfrac{1}{2}(\lambda_1+\lambda_2)e^{i\theta[\lambda_2(1-\alpha_2)+\frac{1}{2}\alpha_2(\lambda_1+\lambda_2)]}\varphi[\tfrac{1}{2}\alpha_2\theta(\lambda_1-\lambda_2)]$$

$$+(i/\alpha_2)e^{i\theta[\lambda_2(1-\alpha_2)+\frac{1}{2}\alpha_2(\lambda_1+\lambda_2)]}\frac{d}{d\theta}\{\varphi[\tfrac{1}{2}\alpha_2\theta(\lambda_1-\lambda_2)]\} .$$

Thus

$$\varphi[\tfrac{1}{2}\theta(\lambda_1-\lambda_2)] = \tfrac{1}{2}(\lambda_1+\lambda_2)e^{\frac{1}{2}i\theta(1-\alpha_1)(\lambda_1-\lambda_2)}\varphi[\tfrac{1}{2}\alpha_1\theta(\lambda_1-\lambda_2)]$$

$$-(i/\alpha_1)e^{\frac{1}{2}i\theta(1-\alpha_1)(\lambda_1-\lambda_2)}\frac{d}{d\theta}\{\varphi[\tfrac{1}{2}\alpha_1\theta(\lambda_1-\lambda_2)]\}$$

$$+[1-\tfrac{1}{2}(\lambda_1+\lambda_2)]e^{\frac{1}{2}i\theta(1-\alpha_2)(\lambda_2-\lambda_1)}\varphi[\tfrac{1}{2}\alpha_2\theta(\lambda_1-\lambda_2)]$$

$$+(i/\alpha_2)e^{\frac{1}{2}i\theta(1-\alpha_2)(\lambda_2-\lambda_1)}\frac{d}{d\theta}\{\varphi[\tfrac{1}{2}\alpha_2\theta(\lambda_1-\lambda_2)]\} .$$

Put $v = \frac{1}{2}\theta(\lambda_1-\lambda_2)$ to yield

$$\varphi(v) = \tfrac{1}{2}(\lambda_1+\lambda_2)e^{iv(1-\alpha_1)}\varphi(\alpha_1 v) + [1-\tfrac{1}{2}(\lambda_1+\lambda_2)]e^{-iv(1-\alpha_2)}\varphi(\alpha_2 v) \tag{3.14}$$

$$-[\tfrac{1}{2}i(\lambda_1-\lambda_2)/\alpha_1]e^{iv(1-\alpha_1)}\frac{d}{dv}[\varphi(\alpha_1 v)]$$

$$+[\tfrac{1}{2}i(\lambda_1-\lambda_2)/\alpha_2]e^{-iv(1-\alpha_2)}\frac{d}{dv}[\varphi(\alpha_2 v)],\quad (\lambda_1 > \lambda_2,\ \lambda_1 \neq 1,\ \lambda_2 \neq 0) .$$

Equations (3.13) and (3.14) are satisfied by the characteristic functions of the transformed limiting distribution functions. The transformations (3.10) and (3.11) imply the new boundary conditions on the distribution function $F(x)$ to be

$$(3.15)\qquad \begin{aligned} & F(x) = 0 \ , \quad x \leq -1 \ ; \\ \text{and} \quad & F(x) = 1 \ , \quad x \geq 1 \ . \end{aligned}$$

Following the procedure of Chapter II, the fundamental inversion theorem (2.22) is applied to equations (3.13) and (3.14). Carrying out the inversion on (3.13) one term at a time, the left member becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(t)[i-e^{-ixt}]/(it)\}dt = F(x) - F(0) \ ,$$

while the terms of the right member become:

$$\text{i)} \quad \tfrac{1}{2}(\lambda_1+\lambda_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(\alpha_1 t)e^{-it(1-\alpha_1)}[1-e^{-ixt}]/(it)\} \, dt$$

$$= \tfrac{1}{2}(\lambda_1+\lambda_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(\alpha_1 t)[-(1-e^{-it(1-\alpha_1)})+(1-e^{-it(x+1-\alpha_1)}]/(it)\} \, dt \ .$$

Putting $\theta = \alpha_1 t$ gives

$$\tfrac{1}{2}(\lambda_1+\lambda_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\varphi(\theta)[-(1-e^{-i\theta(1-\alpha_1)/\alpha_1})+(1-e^{-i\theta(x+1-\alpha_1)/\alpha_1})]/(i\theta)\} \, d\theta$$

$$= \tfrac{1}{2}(\lambda_1+\lambda_2)\{-F[(1-\alpha_1)/\alpha_1]+F(0)+F[(x+1-\alpha_1)/\alpha_1] - F(0)\} \ ,$$

$$= \tfrac{1}{2}(\lambda_1+\lambda_2) \ \{F[(x+1-\alpha_1)/\alpha_1] - F[(1-\alpha_1)/\alpha_1]\} \ .$$

ii) $$[1-\tfrac{1}{2}(\lambda_1+\lambda_2)]\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\varphi(\alpha_2 t)\,e^{it(1-\alpha_2)}[1-e^{-ixt}]/(it)\}\,dt$$

$$= [1-\tfrac{1}{2}(\lambda_1+\lambda_2)]\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\varphi(\alpha_2 t)[-(1-e^{it(1-\alpha_2)})+(1-e^{-it(x-1+\alpha_2)})]/(it)\}dt,$$

$$= [1-\tfrac{1}{2}(\lambda_1+\lambda_2)]\,\{F[(x-1+\alpha_2)/\alpha_2] - F[(\alpha_2-1)/\alpha_2]\}\ .$$

iii) $$\frac{i(\lambda_2-\lambda_1)}{2\alpha_1}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{d}{dt}\,[\varphi(\alpha_1 t)]e^{-it(1-\alpha_1)}[1-e^{-ixt}]/(it)\}dt\ ,$$

which after integrating by parts becomes

$$-\frac{i(\lambda_2-\lambda_1)}{2\alpha_1}\Big\{[\varphi(\alpha_1 t)(e^{-it(1-\alpha_1)}-e^{-it(x+1-\alpha_1)})/(it)]_{t=-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty}\varphi(\alpha_1 t)\ \{i[(1-\alpha_1)e^{-it(1-\alpha_1)}-(x+1-\alpha_1)e^{-it(x+1-\alpha_1)}]/(it)$$

$$+\ [e^{-it(1-\alpha_1)}-e^{-it(x+1-\alpha_1)}]/(it^2)\}\ dt\Big\}\ .$$

Since $|\varphi(t)|\leq 1$, the integrated term vanishes, leaving

$$\frac{\lambda_2-\lambda_1}{2\alpha_1}\,\frac{1}{2\pi}\Big\{(1-\alpha_1)\int_{-\infty}^{\infty}\{\varphi(\alpha_1 t)[e^{-it(1-\alpha_1)}-e^{-it(x+1-\alpha_1)}]/(it)\}\ dt$$

$$-\int_{-\infty}^{\infty}\{\varphi(\alpha_1 t)[(xe^{-it(x+1-\alpha_1)})/(it)\ +(e^{-it(1-\alpha_1)}-e^{-it(x+1-\alpha_1)})/t^2]\ dt\Big\}\ ,$$

$$= \frac{\lambda_2-\lambda_1}{2\alpha_1}\Big\{\frac{1-\alpha_1}{2\pi}\int_{-\infty}^{\infty}\{\varphi(\alpha_1 t)[-(1-e^{-it(1-\alpha_1)})+(1-e^{-it(x+1-\alpha_1)})]/(it)\}dt$$

$$-\ h(x)\Big\}\ ,$$

$$= \frac{\lambda_2-\lambda_1}{2\alpha_1}\left\{(1-\alpha_1)\{F[(x+1-\alpha_1)/\alpha_1] - F[(1-\alpha_1)/\alpha_1]\} - h(x)\right\} ,$$

where $h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\left\{\varphi(\alpha_1 t)[(xe^{-it(x+1-\alpha_1)})/(it)+(e^{-it(1-\alpha_1)}-e^{-it(x+1-\alpha_1)})/t^2]\right\}dt .$

Since $h(x)$ is uniformly convergent in x , differentiation with respect to x may be carried out under the integral sign.

Thus

$$h'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\alpha_1 t)\left[\frac{e^{-it(x+1-\alpha_1)}}{it} - xe^{-it(x+1-\alpha_1)} + \frac{ie^{-it(x+1-\alpha_1)}}{t}\right] dt ,$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\alpha_1 t)(xe^{-it(x+1-\alpha_1)})\, dt ,$$

$$= -\frac{x}{2\pi} \int_{-\infty}^{\infty} \varphi(\alpha_1 t)\, e^{-it(x+1-\alpha_1)}\, dt .$$

Put $\theta = \alpha_1 t$ so that

$$h'(x) = -\left(\frac{x}{\alpha_1}\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\theta)e^{-i\theta(x+1-\alpha_1)/\alpha_1}\, d\theta .$$

For those valuesof x for which $F(x)$ has a derivative $f(x)$, the fundamental inversion theorem gives

$$(3.16) \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t)e^{-ixt}\, dt .$$

Applying (3.16) to $h'(x)$ gives

$$h'(x) = -(x/\alpha_1)f[(x+1-\alpha_1)/\alpha_1] .$$

Integrating to evaluate $h(x)$ we obtain

$$h(x) = -\frac{1}{\alpha_1} \int_{-\infty}^{x} tf[(t+1-\alpha_1)/\alpha_1] \, dt + K_1 ,$$

where K_1 is a constant of integration.

Making a change of variable

$$h(x) = - \int_{-\infty}^{(x+1-\alpha_1)/\alpha_1} f(u)[\alpha_1 u-1+\alpha_1)] \, du + K_1 ,$$

$$= (1-\alpha_1)F[(x+1-\alpha_1)/\alpha_1] - \alpha_1 \int_{-\infty}^{(x+1-\alpha_1)/\alpha_1} uf(u)du + K_1 ,$$

or

$$h(x) = (1-\alpha_1) F[(x+1-\alpha_1)/\alpha_1] - \alpha_1\{((x+1-\alpha_1)/\alpha_1]F[(x+1-\alpha_1)/\alpha_1]\}$$

$$- \int_{-\infty}^{(x+1-\alpha_1)/\alpha_1} F(u)du\} + K_1 + K_2 ,$$

where K_2 is another constant of integration.

The total contribution from iii) is

$$\frac{\lambda_2-\lambda_1}{2\alpha_1} \left\{(1-\alpha_1)\{F[(x+1-\alpha_1)/\alpha_1] - F[(1-\alpha_1)/\alpha_1]\} - (1-\alpha_1)F[(x+1-\alpha_1)/\alpha_1]\right.$$

$$\left. + (x+1-\alpha_1) F[(x+1-\alpha_1)/\alpha_1]-\alpha_1 \int_{-\infty}^{(x+1-\alpha_1)/\alpha_1} F(u)du + K_1 + K_2\right\}$$

$$= \frac{\lambda_2-\lambda_1}{2\alpha_1}\Big\{(x+1-\alpha_1)F[(x+1-\alpha_1)/\alpha_1] - (1-\alpha_1)F[(1-\alpha_1)/\alpha_1]$$

$$- \alpha_1\int_{-\infty}^{(x+1-\alpha_1)/\alpha_1} F(u)\,du + K_1 + K_2\Big\} .$$

iv) $$\frac{i(\lambda_2-\lambda_1)}{2\alpha_2}\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{d}{dt}[\varphi(\alpha_2 t)]\,e^{it(1-\alpha_2)}(1-e^{-ixt})/(it)\}dt .$$

Integrating by parts as before yields

$$\frac{i(\lambda_2-\lambda_1)}{2\alpha_2}\ \frac{1}{2\pi}\Big\{[\varphi(\alpha_2 t)(e^{it(1-\alpha_2)}-e^{-it(x-1+\alpha_2)})/(it)]_{t=-\infty}^{\infty}$$

$$-\int_{-\infty}^{\infty}\varphi(\alpha_2 t)\ \{i[(i-\alpha_2)e^{it(1-\alpha_2)}+(x-1+\alpha_2)e^{-it(x-1+\alpha_2)}]/(it)$$

$$-[e^{it(1-\alpha_2)}-e^{-it(x-1+\alpha_2)}]/(it^2)\}dt\Big\} .$$

The integrated part vanishes leaving

$$\frac{\lambda_2-\lambda_1}{2\alpha_2}\Big\{\frac{1-\alpha_2}{2\pi}\int_{-\infty}^{\infty}\{\varphi(\alpha_2 t)[e^{it(1-\alpha_2)}-e^{-it(x-1+\alpha_2)}]/(it)\}\,dt$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_2 t)[(xe^{-it(x-1+\alpha_2)})/(it)+(e^{it(1-\alpha_2)}-e^{-it(x-1+\alpha_2)})/t^2]dt\Big\}$$

$$= \frac{\lambda_2-\lambda_1}{2\alpha_2}\Big\{(1-\alpha_2)\{F[(x-1+\alpha_2)/\alpha_2] - F[(\alpha_2-1)/\alpha_2]\} + g(x)\Big\} ,$$

where

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_2 t)[(xe^{-it(x-1+\alpha_2)})/(it)+(e^{it(1-\alpha_2)}-e^{-it(x-1+\alpha_2)})/t^2]dt .$$

$$g'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\alpha_2 t)\left[\frac{e^{-it(x-1+\alpha_2)}}{it} - xe^{-it(x-1+\alpha_2)} - \frac{e^{-it(x-1+\alpha_2)}}{it}\right] dt ,$$

$$= \frac{-x}{2\pi} \int_{-\infty}^{\infty} \varphi(\alpha_2 t) e^{-it(x-1+\alpha_2)} dt ,$$

$$= -(x/\alpha_2) f[(x-1+\alpha_2)/\alpha_2] \quad \text{(at the points where } f \text{ exists).}$$

Therefore

$$g(x) = -\frac{1}{\alpha} \int_{-\infty}^{x} uf[(u-1+\alpha_2)/\alpha_2]\, du + K_3 ,$$

$$g(x) = -\int_{-\infty}^{(x-1+\alpha_2)/\alpha_2} (\alpha_2 v + 1 - \alpha_2) f(v)\, dv + K_3 ,$$

$$= K_3 - (1-\alpha_2) F[(x-1+\alpha_2)/\alpha_2] - \alpha_2 \int_{-\infty}^{(x-1+\alpha_2)/\alpha_2} vf(v)\, dv ,$$

$$= K_3 - (1-\alpha_2) F[(x-1+\alpha_2)/\alpha_2] - \alpha_2 \Big\{ ((x-1+\alpha_2)/\alpha_2) F[(x-1+\alpha_2)/\alpha_2]$$

$$- \int_{-\infty}^{(x-1+\alpha_2)/\alpha_2} F(v)\, dv \Big\} + K_4 .$$

The inverted form of iv) is

$$\frac{\lambda_2-\lambda_1}{2\alpha_2} \{(1-\alpha_2)[F[(x-1+\alpha_2)/\alpha_2] - F[(\alpha_2-1)/\alpha_2]] + K_3 + K_4 - (1-\alpha_2) F[(x-1+\alpha_2)/\alpha_2]$$

$$-(x-1+\alpha_2) F[(x-1+\alpha_2)/\alpha_2] + \alpha_2 \int_{-\infty}^{(x-1+\alpha_2)/\alpha_2} F(v)\, dx\} ,$$

$$= \frac{\lambda_2-\lambda_1}{2\alpha_2} \{-(x-1+\alpha_2) F[(x-1+\alpha_2)/\alpha_2] - (1-\alpha_2) F[(\alpha_2-1)/\alpha_2]$$

$$+ \alpha_2 \int_{-\infty}^{(x-1+\alpha_2)/\alpha_2} F(v) dv + K_3 + K_4\},$$ (where K_3 and K_4 are constants of integration).

Collecting terms of the inverted functional equation, we obtain

$$F(x)-F(0) = \tfrac{1}{2}(\lambda_2-\lambda_1)\{F[(x+1-\alpha_1)/\alpha_1]-F[(1-\alpha_1)/\alpha_1]\}$$

$$+ [1-\tfrac{1}{2}(\lambda_1+\lambda_2)]\ \{F[(x-1+\alpha_2)/\alpha_2]-F[(\alpha_2-1)/\alpha_2]\}$$

$$+ [(\lambda_2-\lambda_1)/(2\alpha_1)]\Big\{(x+1-\alpha_1)F[(x+1-\alpha_1)/\alpha_1]-(1-\alpha_1)F[(1-\alpha_1)/\alpha_1]$$

$$-\alpha_1\int_{-\infty}^{(x+1-\alpha_1)/\alpha_1}F(u)\ du\ +K_1+K_2\Big\}$$

$$+ [(\lambda_2-\lambda_1)/(2\alpha_2)]\Big\{-(x-1+\alpha_2)F[(x-1+\alpha_2)/\alpha_2]-(1-\alpha_2)F[(\alpha_2-1)/\alpha_2]$$

$$+\alpha_2\int_{-\infty}^{(x-1+\alpha_2)/\alpha_2}F(u)\ du+K_3+K_4\Big\}\ ,$$

or,

$$F(x)-F(0) = \tfrac{1}{2}[\lambda_1+\lambda_2+(\lambda_2-\lambda_1)(x+1-\alpha_1)/\alpha_1]F[(x+1-\alpha_1)/\alpha_1]$$

$$+ [1-\tfrac{1}{2}(\lambda_1+\lambda_2)-\tfrac{1}{2}(\lambda_2-\lambda_1)(x-1+\alpha_2)(x-1+\alpha_2)/\alpha_2]F[(x-1+\alpha_2)/\alpha_2]$$

$$- \tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-\infty}^{(x+1-\alpha_1)/\alpha_1}F(u)du+\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-\infty}^{(x-1+\alpha_2)/\alpha_2}F(v)\ dv$$

$$- \tfrac{1}{2}[(\lambda_1+\lambda_2)-(\lambda_2-\lambda_1)(1-\alpha_1)/\alpha_1]F[(1-\alpha_1)/\alpha_1]$$

$$- [1-\tfrac{1}{2}(\lambda_1+\lambda_2)\ +\ \tfrac{1}{2}(\lambda_2-\lambda_1)(1-\alpha_2)/\alpha_2]F[(\alpha_2-1)/\alpha_2]$$

$$+ \tfrac{1}{2}(\lambda_2-\lambda_1)(K_1+K_2)/\alpha_1\ +\ \tfrac{1}{2}(\lambda_2-\lambda_1)(K_3+K_4)/\alpha_2\ .$$

Imposing the condition that $F(x) = 0,\ x \leq -1$ gives

$$0 = F(0)\ -\ \tfrac{1}{2}[\lambda_1+\lambda_2+(\lambda_2-\lambda_1)(1-\alpha_1)/\alpha_1]F[(1-\alpha_1)/\alpha_1]$$

$$- [1-\tfrac{1}{2}(\lambda_1+\lambda_2)\ +\ \tfrac{1}{2}(\lambda_2-\lambda_1)(1-\alpha_2)/\alpha_2]F[(\alpha_2-1)/\alpha_2]$$

$$+ \tfrac{1}{2}(\lambda_2-\lambda_1)(K_1+K_2)/\alpha_1\ +\ \tfrac{1}{2}(\lambda_2-\lambda_1)(K_3+K_4)/\alpha_2\ .$$

Thus the limiting distribution functions corresponding to the characteristic functional equation (3.13) satisfy both the following functional integral equation

$$(3.17) \qquad F(x) = [\lambda_1+(\lambda_2-\lambda_1)(x+1)/(2\alpha_1)]F[(x+1-\alpha_1)/\alpha_1]$$

$$+ [1-\lambda_2-(x-1)(\lambda_2-\lambda_1)/(2\alpha_2)]F[(x-1+\alpha_2)/\alpha_2]$$

$$- \tfrac{1}{2}(\lambda_2-\lambda_1)\int_{(x-1+\alpha_2)/\alpha_2}^{(x+1-\alpha_1)/\alpha_1} F(u)\,du \ , \qquad (\lambda_2 > \lambda_1)$$

and the auxiliary conditions

$$F(x) = 0 \ , \quad x \leq -1 \ ; \qquad F(x) = 1 \ , \quad x \geq 1 \ ;$$

$$0 < \alpha_1 \ , \ \alpha_2 < 1 \qquad ; \qquad 0 \leq \lambda_1 < \lambda_2 < 1 \quad .$$

Exactly the same procedure is used to invert the characteristic functional equation (3.14) to determine the equation satisfied by the corresponding distribution functions. Under the inversion, the left member becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \{\varphi(t)[1 - e^{-ixt}]\,/(it)\}\,dt = F(x) - F(0) \ .$$

The terms of the right side of (3.14) become

i) $\tfrac{1}{2}(\lambda_1+\lambda_2)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \{\varphi(\alpha_1 t)\ e^{it(1-\alpha_1)}[1-e^{-ixt}]/(it)\}\,dt$

$$= \tfrac{1}{2}(\lambda_1+\lambda_2)[F[(x-1+\alpha_1)/\alpha_1] - F[(\alpha_1-1)/\alpha_1]] \ .$$

ii) $[1-\tfrac{1}{2}(\lambda_1+\lambda_2)]\frac{1}{2\pi}\int_{-\infty}^{\infty} \{\varphi(\alpha_2 t)\ e^{-it(1-\alpha_2)}(1-e^{-ixt})/(it)\}\,dt$

$$= [1-\tfrac{1}{2}(\lambda_1+\lambda_2)][F[(x+1-\alpha_2)/\alpha_2] - F[(1-\alpha_2)/\alpha_2]] \;.$$

iii) $$\frac{-i(\lambda_1-\lambda_2)}{2\alpha_1}\;\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{d}{dt}\,[\varphi(\alpha_1 t)]e^{it(1-\alpha_1)}[1-e^{-ixt}]/(it)\right\}dt$$

Integrating by parts gives

$$\frac{-i(\lambda_1-\lambda_2)}{2\alpha_1}\left\{\frac{1}{2\pi}\,[\varphi(\alpha_1 t)(e^{it(1-\alpha_1)}-e^{-it(x-1+\alpha_1)})/(it)]_{T=-\infty}^{\infty}\right.$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_1 t)\left\{[(1-\alpha_1)e^{it(1-\alpha_1)}+(x-1+\alpha_1)e^{-it(x-1+\alpha_1)}]/t\right.$$

$$\left.\left.-[e^{it(1-\alpha_1)}-e^{-it(x-1+\alpha_1)}]/(it^2)\right\}dt\right\} ,$$

$$= \frac{\lambda_1-\lambda_2}{2\alpha_1}\left\{-\frac{1-\alpha_1}{2\pi}\int_{-\infty}^{\infty}\left\{\varphi(\alpha_1 t)[e^{it(1-\alpha_1)}-e^{-it(x-1+\alpha_1)}]/(it)\right\}dt\right.$$

$$\left.+\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_1 t)\left\{(-xe^{-it(x-1+\alpha_1)})/(it)-(e^{-it(1-\alpha_1)}-e^{-it(x-1+\alpha_1)})/t^2\right\}dt\right\},$$

$$= \frac{\lambda_1-\lambda_2}{2\alpha_1}\left\{-(1-\alpha_1)[F[(x-1+\alpha_1)/\alpha_1]-F[(\alpha_1-1)/\alpha_1]] + h(x)\right\} .$$

where $$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_1 t)\left\{-[xe^{-it(x-1+\alpha_1)}]/(it)-[e^{it(1-\alpha_1)}-e^{-it(x-1+\alpha_1)}]/t^2\right\}dt.$$

$$h'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_1 t)\left\{-\frac{e^{-it(x-1+\alpha_1)}}{it}+xe^{-it(x-1+\alpha_1)}-\frac{ie^{-it(x-1+\alpha_1)}}{t}\right\}dt ,$$

$$= \frac{x}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_1 t)\,e^{-it(x-1+\alpha_1)}dt ,$$

$$= (x/\alpha_1)\,f[(x-1+\alpha_1)/\alpha_1] ,$$ provided f exists.

Therefore

$$h(x) = \frac{1}{\alpha}\int_{-\infty}^{x} uf[(u-1+\alpha_1)/\alpha_1]\,du + K_1\,,$$ where K_1 is a constant

of integration.

$$h(x) = (1-\alpha_1)F[(x-1+\alpha_1)/\alpha_1] + \alpha_1\int_{-\infty}^{(x-1+\alpha_1)/\alpha_1} v\,f(v)\,dv + K_1\,,$$

$$= (1-\alpha_1)F[(x-1+\alpha_1)/\alpha_1] + (x-1+\alpha_1)F[(x-1+\alpha_1)/\alpha_1]$$

$$-\alpha_1\int_{-\infty}^{(x-1+\alpha_1)/\alpha_1} F(v)\,dv] + K_1 + K_2\,.$$

Thus the inverted form of iii) is

$$\frac{\lambda_1-\lambda_2}{2\alpha_1}\Big\{(x-1+\alpha_1)F[(x-1+\alpha_1)/\alpha_1] + (1-\alpha_1)F[(\alpha_1-1)/\alpha_1] - \alpha_1\int_{-\infty}^{(x-1+\alpha_1)\,\alpha_1}\Big(F(u)\Big)du + K_1 + K_2\Big\}\,.$$

iv) $$\frac{i(\lambda_1-\lambda_2)}{2\alpha_2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\frac{d}{dt}\,[\varphi(\alpha_2 t)]e^{-it(1-\alpha_2)}[1-e^{-ixt}]/(it)\Big\}\,dt\,.$$

Again integrating by parts we obtain

$$\frac{i(\lambda_1-\lambda_2)}{2\alpha_2}\Big\{\frac{1}{2\pi}\,[\varphi(\alpha_2 t)(e^{-it(1-\alpha_2)}-e^{-it(x+1-\alpha_2)})/(it)]_{t=-\infty}^{\infty}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_2 t)\Big\{[(1-\alpha_2)e^{-it(1-\alpha_2)} - (x+1-\alpha_2)e^{-it(x+1-\alpha_2)}]/t$$

$$+[e^{-it(1-\alpha_2)}-e^{-it(x+1-\alpha_2)}]/(it^2)\Big\}\,dt\Big\}\,,$$

$$= \frac{\lambda_1-\lambda_2}{2\alpha_2}\left\{-\frac{1-\alpha_2}{2\pi}\int_{-\infty}^{\infty}\left\{\varphi(\alpha_2 t)[e^{-it(1-\alpha_2)}-e^{-it(x+1-\alpha_2)}]/(it)\right\}dt\right.$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\varphi(\alpha_2 t)[(xe^{-it(x+1-\alpha_2)})/(it)+(e^{-it(1-\alpha_2)}\right.$$

$$\left.\left.-e^{-it(x+1-\alpha_2)})/t^2\right\}dt\right\},$$

$$= \frac{\lambda_1-\lambda_2}{2\alpha_2}\left\{-(1-\alpha_2)\left[F[(x+1-\alpha_2)/\alpha_2]-F[(1-\alpha_2)/\alpha_2]\right\}+g(x)\right],$$

where $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_2 t)[(xe^{-it(x+1-\alpha_2)})/(it)+(e^{-it(1-\alpha_2)}-e^{-it(x+1-\alpha_2)})/t^2]\,dt.$

$$g'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_2 t)\left[\frac{e^{-it(x+1-\alpha_2)}}{it}-xe^{-it(x+1-\alpha_2)}-\frac{e^{-it(x+1-\alpha_2)}}{it}\right]dt$$

$$= -\frac{x}{2\pi}\int_{-\infty}^{\infty}\varphi(\alpha_2 t)e^{-it(x+1-\alpha_2)}\,dt,$$

$$= -(x/\alpha_2)\,f[(x+1-\alpha_2)/\alpha_2]\quad,$$ (if the derivative $f(x)$ of $F(x)$ exists).

Thus $g(x) = -\frac{1}{\alpha_2}\int_{-\infty}^{x} uf[(u+1-\alpha_2)/\alpha_2]\,du + K_3$, ($K_3$ is a constant of integration),

$$(g(x)) = (1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2] - \alpha_2\int_{-\infty}^{(x+1-\alpha_2)/\alpha_2} uF(u)\,du + K_3,$$

$$= (1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2] - (x+1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2]$$

$$+ \alpha_2\int_{-\infty}^{(x+1-\alpha_2)/\alpha_2} F(u)\,du + K_3 + K_4 .$$

Thus the inverted term is

$$\frac{\lambda_1-\lambda_2}{2\alpha_2}\Big\{-(1-\alpha_2)\{F[(x+1-\alpha_2)/\alpha_2]-F[(1-\alpha_2)/\alpha_2]\}+(1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2]$$

$$-(x+1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2]+\alpha_2\int_{-\infty}^{(x+1-\alpha_2)/\alpha_2} F(u)\,du + K_3 + K_4 \;.$$

$$= \frac{\lambda_1-\lambda_2}{2\alpha_2}\Big\{(1-\alpha_2)F[(1-\alpha_2)/\alpha_2] - (x+1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2]$$

$$+ \alpha_2\int_{-\infty}^{(x+1-\alpha_2)/\alpha_2} F(u)\,du + K_3 + K_4 \;.$$

Collecting the terms of the inverted equation we obtain

$$F(x) - F(0) = \tfrac{1}{2}(\lambda_1+\lambda_2)\ \{F[(x-1+\alpha_1)/\alpha_1] - F[(\alpha_1-1)/\alpha_1]\}$$

$$+ [1-\tfrac{1}{2}(\lambda_1+\lambda_2)]\ \{F[(x+1-\alpha_2)/\alpha_2] - F[(1-\alpha_2)/\alpha_2]\}$$

$$+ \frac{\lambda_1-\lambda_2}{2\alpha_1}\Big\{(x-1+\alpha_1)F[(x-1+\alpha_1)/\alpha_1] + (1-\alpha_1)F[(\alpha_1-1)/\alpha_1]$$

$$- \alpha_1\int_{-\infty}^{(x-1+\alpha_1)/\alpha_1} F(u)\,du + K_1 + K_2\Big\}$$

$$+ \frac{\lambda_1-\lambda_2}{2\alpha_2}\Big\{(1-\alpha_2)F[(1-\alpha_2)/\alpha_2] - (x+1-\alpha_2)F[(x+1-\alpha_2)/\alpha_2]$$

$$+ \alpha_2\int_{-\infty}^{(x-1+\alpha_2)/\alpha_2} F(u)\,du + K_3 + K_4\Big\}$$

Imposing the condition $F(x) = 0$, $x \leq -1$ gives

$$F(0) = \tfrac{1}{2}[\lambda_1+\lambda_2 - (\lambda_1-\lambda_2)(1-\alpha_1)/\alpha_1]F[(\alpha_1-1)/\alpha_1]$$

$$- [1-\tfrac{1}{2}(\lambda_1+\lambda_2) - \tfrac{1}{2}(\lambda_1-\lambda_2)(1-\alpha_2)/\alpha_2]F[(1-\alpha_2)/\alpha_2]$$

$$+ \tfrac{1}{2}(\lambda_1-\lambda_2)[(K_1+K_2)/\alpha_1 + (K_3+K_4)/\alpha_2] \;.$$

Thus for $\lambda_1 > \lambda_2$ the limiting distributions satisfy the functional equation

$$(3.18) \qquad F(x) = [\lambda_1+\tfrac{1}{2}(\lambda_1-\lambda_2)(x-1)/\alpha_1]F[(x-1+\alpha_1)/\alpha_1]$$
$$+ [1-\lambda_2-\tfrac{1}{2}(\lambda_1-\lambda_2)(x+1)/\alpha_2]F[(x+1-\alpha_2)/\alpha_2]$$
$$+ \tfrac{1}{2}(\lambda_1-\lambda_2)\int_{(x-1+\alpha_1)/\alpha_1}^{(x+1-\alpha_2)/\alpha_2} F(u)du, \quad (\lambda_1 > \lambda_2,\ \lambda_1 \neq 1,\ \lambda_2 \neq 0),$$

and the auxiliary conditions

$$F(x) = 0, \quad x \leq -1 ; \qquad F(x) = 1, \quad x \geq 1 ;$$
$$0 < \alpha_1, \alpha_2 < 1 \quad ; \qquad 0 \leq \lambda_2 < \lambda_1 \leq 1 .$$

§ 3.4 Additional Properties of the Limiting Distributions

The characteristic functions of the untransformed limiting distribution functions satisfy the equation

$$\varphi(\theta) = e^{i\lambda_2(1-\alpha_2)\theta}\varphi(\alpha_2\theta)-(i/\alpha_1)e^{i\lambda_1(1-\alpha_1)\theta}\frac{d}{d\theta}[\varphi(\alpha_1\theta)]$$
$$+ (i/\alpha_2)e^{i\lambda_2(1-\alpha_2)\theta}\frac{d}{d\theta}[\varphi(\alpha_2\theta)].$$

This equation may be re-written in the form

$$(3.19) \qquad \varphi(\theta) = e^{i\lambda_2(1-\alpha_2)\theta}\varphi(\alpha_2\theta)-ie^{i\lambda_1(1-\alpha_1)\theta}\varphi'(\alpha_1\theta)+ie^{i\lambda_2(1-\alpha_2)\theta}\varphi'(\alpha_2\theta) ,$$

where $\varphi'(\alpha_1\theta) = \dfrac{d}{d(\alpha_1\theta)}[\varphi(\alpha_1\theta)]$, $\varphi'(\alpha_2\theta) = \dfrac{d}{d(\alpha_2\theta)}[\varphi(\alpha_2\theta)]$.

Since the characterstic functions have the property that $\varphi(0) = 1$ and

$$V_n = (1/i^n)\ \frac{d^n}{d\theta^n}[\varphi(\theta)]_{\theta=0}$$

where V_n is the n^{th} moment of the desired distribution function, we

seek to derive the moments of the limiting distribution from the characteristic equation (3.19). For $\theta = 0$, equation (3.19) reduces to

$$0 = (-i + i)\,\varphi'(0)\ .$$

Thus $\varphi'(0)$ is indeterminate.

Differentiating (3.19) with respect to θ gives

$$\begin{aligned}\varphi'(\theta) = {} & i\lambda_2(1-\alpha_2)e^{i\lambda_2(1-\alpha_2)\theta}\varphi(\alpha_2\theta) + [\alpha_2-\lambda_2(1-\alpha_2)]e^{i\lambda_2(1-\alpha_2)\theta}\varphi'(\alpha_2\theta) \\ & + \lambda_1(1-\alpha_1)e^{i\lambda_1(1-\alpha_1)\theta}\varphi'(\alpha_1\theta) - i\alpha_1 e^{i\lambda_1(1-\alpha_1)\theta}\varphi''(\alpha_1\theta) \\ & + i\alpha_2 e^{i\lambda_2(1-\alpha_2)\theta}\varphi''(\alpha_2\theta)\ .\end{aligned}$$

For $\theta = 0$ the above equation becomes

$$-i\lambda_2(1-\alpha_2) = [\alpha_2 - \lambda_2(1-\alpha_2) + \lambda_1(1-\alpha_1) - 1]\varphi'(0) + i(\alpha_2-\alpha_1)\varphi''(0)\ .$$

This equation cannot be solved in general since there are two unknowns $\varphi'(0)$ and $\varphi''(0)$. However, if $\alpha_1 = \alpha_2$, the coefficient of $\varphi''(0)$ vanishes. Thus replacing α_1 and α_2 by α, we have

$$\varphi'(0) = i\lambda_2/(1 + \lambda_2 - \lambda_1)\ ,\quad (\alpha_1 = \alpha_2 = \alpha)\ .$$

<u>Lemma 3.1</u> The functional differential equation satisfied by the characteristic functions of the limiting distribution of response probabilities specifies unique moments for $\alpha_1 = \alpha_2 = \alpha$.

<u>Proof</u> Imposing the equal alpha condition on (3.19) gives

$$\varphi(\theta) = e^{i\lambda_2(1-\alpha)\theta}\varphi(\alpha\theta) + i[e^{i\lambda_2(1-\alpha)\theta} - e^{i\lambda_1(1-\alpha)\theta}]\varphi'(\alpha\theta)\ . \tag{3.20}$$

Differentiating (3.20) n times with respect to θ then imposing $\theta = 0$ causes the coefficient of $\varphi^{(n+1)}(0)$ to vanish. This leaves $\varphi^{(n)}(0)$ as a function of $\varphi^{(k)}(0)$ $(k=0,1,\ldots,n-1)$. Thus the moments $\{V_n\}$ may be calculated in order of increasing n.

Direct verification shows that the result of lemma 3.1 also applies to the transformed characteristic equations (3.13) and (3.14).

It has been shown above that the limiting distributions of the transformed response probabilities of the Subject-Controlled Model satisfy either

$$(3.21)\qquad \begin{aligned} F(x) &= [\lambda_1 + (x+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F[(x+1-\alpha_1)/\alpha_1] \\ &+ [1-\lambda_2 + (1-x)(\lambda_2-\lambda_1)/(2\alpha_2)]F[(x-1+\alpha_2)/\alpha_2] \\ &- \tfrac{1}{2}(\lambda_2-\lambda_1)\int_{(x-1+\alpha_2)/\alpha_2}^{(x+1-\alpha_1)/\alpha_1} F(u)\,du\ , \qquad (\lambda_2 > \lambda_1)\ ; \end{aligned}$$

or

$$(3.22)\qquad \begin{aligned} F(x) &= [\lambda_1 + (x-1)(\lambda_1-\lambda_2)/(2\alpha_1)]F[(x-1+\alpha_1)/\alpha_1] \\ &+ [1-\lambda_2 - (x+1)(\lambda_1-\lambda_2)/(2\alpha_2)]F[(x+1-\alpha_2)/\alpha_2] \\ &+ \tfrac{1}{2}(\lambda_1-\lambda_2)\int_{(x-1+\alpha_1)/\alpha_1}^{(x+1-\alpha_2)/\alpha_2} F(u)\,du\ , \qquad (\lambda_1 > \lambda_2, \lambda_1 \neq 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda_2 \neq 0)\ ; \end{aligned}$$

with the auxiliary conditions

$$(3.24)\qquad \begin{aligned} &F(x) = 0\ ,\quad x \leq -1\ ;\quad F(x) = 1\ ,\quad x \geq 1\ ; \\ &0 < \alpha_1\ ,\ \alpha_2 < 1\ ;\qquad 0 \leq \lambda_1\ ,\ \lambda_2 \leq 1\ . \end{aligned}$$

Theorem 3.1 The limiting distribution functions satisfying the functional systems (3.21),(3.23),(3.24) and (3.22),(3.23),(3.24) are unique if $\alpha_1=\alpha_2=\alpha$.

Proof. Lemma 3.1 proves that the characteristic functions are unique for $\alpha_1=\alpha_2=\alpha$. Thus by the uniqueness theorem of characteristic functions, there can be only one distribution function that satisfies the above functional systems (c.f. Theorem 2.3).

Theorem 3.2 For $\alpha_1+\alpha_2<1$, the distribution functions satisfying the above functional systems are purely singular functions which are constant everywhere except perhaps on a non-denumerable set of points of Lebesgue measure zero.

Proof. The arguments of (3.21) and (3.22) are of the form

$$(x+1-\alpha')/\alpha' \ , \quad x, \quad (x-1+\alpha^*)/\alpha^* \ .$$

(For (3.21) read α_1 for α' and α_2 for α^*, for (3.22) read α_2 for α' and α_1 for α^*). In the domain (-1,1) the arguments are ordered from largest to smallest as written above. For $\alpha_1+\alpha_2<1$, we know from theorem 2.1 that there exists an interval $I_{1,1}=[2\alpha'-1,1-2\alpha^*]$ such that $(x+1-\alpha')/\alpha' \geq 1$ and $(x-1+\alpha^*)/\alpha^* \leq -1$ simultaneously.

Consider (3.21) first. Letting $x\in I_{1,1}=[2\alpha_1-1,1-2\alpha_2]$ and applying the boundary conditions (3.23) gives

$$F(x)=\lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{(x+1-\alpha_1)/\alpha_1} F(u)\,du \ ,$$

$$=\lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)-\tfrac{1}{2}(\lambda_2-\lambda_1)\left[\int_{-1}^{1} F(u)du+\int_{1}^{(x+1-\alpha_1)/\alpha_1} du\right] .$$

Therefore
$$F(x)=\lambda_2-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{1} F(u)\,du \ , \quad (x\in I_{1,1}) \ .$$

The above equation shows that F(x) is constant for all $x \in I_{1,1}$. Let this constant be $F_{1,1}$ so that

$$F_{1,1}=\lambda_2-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{1} F(u)\,du \ . \tag{3.25}$$

Since the value of $\int_{-1}^{1} F(u)\,du$ is unknown, (3.25) does not yield

the explicit value of $F_{1,1}$ in general.

Using the notation introduced in Chapter II (p. 24 f.f),let $G_{1,1}$ denote the gap $(-1, 2\alpha_1-1)$, and $G_{1,2}$ denote the gap $(1 - 2\alpha_2, 1)$. From the interval $I_{1,1}$, two new disjoint intervals where the distribution function takes a constant value may be found by the operators B_1 (2.37) and B_2(2.38). The value of the distribution function on the interval $I_{2,1} = [\alpha_1(2\alpha_1-1)-(1-\alpha_1), \alpha_1(1-2\alpha_2)-(1-\alpha_1)]$ is given by

$$F(x) = [\lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F_{1,1} - \tfrac{1}{2}(\lambda_2-\lambda_1)\left[\int_{-1}^{2\alpha_1-1} F(u)\,du + \int_{2\alpha_1-1}^{(x+1-\alpha_1)/\alpha_1} F_{1,1}\,du\right],$$

$$(x \in I_{2,1}).$$

Simplifying, we have

$$F(x) = [\lambda_1 + \alpha_1(\lambda_2-\lambda_1)]F_{1,1} - \tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{2\alpha_1-1} F(u)\,du \quad .$$

Since the solution on $I_{2,1}$ is also a constant, let $F_{2,1} = F(x),(x \in I_{2,1})$ so that

$$F_{2,1} = [\lambda_1 + \alpha_1(\lambda_2-\lambda_1)]F_{1,1} - \tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{2\alpha_1-1} F(u)\,du \;. \tag{3.26}$$

Similarly, the distribution function on the interval

$$I_{2,2} = [\alpha_2(2\alpha_1-1) + (1-\alpha_2), \alpha_2(1-2\alpha_2) + (1-\alpha_2)]$$

is given by $F(x) = F_{2,2}$, $(x \in I_{2,2})$ where

$$F_{2,2} = \lambda_2 + [1-\lambda_2 + \alpha_2(\lambda_2-\lambda_1)]F_{1,1} - \tfrac{1}{2}(\lambda_2-\lambda_1)\int_{1-2\alpha_2}^{1} F(u)\,du \;. \tag{3.27}$$

In this problem a set of disjoint intervals over which the distribution function is constant is generated in exactly the same manner as the intervals of the Experimenter Model. Thus the intervals are disjoint and may be generated in a series of stages such that at the n^{th} stage 2^{n-1} new intervals are generated from the 2^{n-2} intervals of the $n-1$ stage by the transformations B_1 and B_2. We now proceed by induction to show that the value of the distribution function in the interval $I_{n,i}$ is given by

$$F_{n,i} = [\lambda_1+\tfrac{1}{2}(\lambda_2-\lambda_1)(1+L_{n-1,i})]F_{n-1,i}-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{L_{n-1,i}} F(u)\,du\,, \tag{3.28}$$

$$(n=2,3,4,\ldots,\quad i = 1,2,\ldots,2^{n-2})$$

or by

$$F_{n,i} = \lambda_2 + [1-\lambda_2-\tfrac{1}{2}(\lambda_2-\lambda_1)(U_{n-1,i-2^{n-2}}-1)F_{n-1,i-2^{n-2}} -\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{U_{n-1,i-2^{n-2}}}^{1} F(u)\,du\,, \quad (n=2,3,\ldots,i=2^{n-2}+1,\ 2^{n-2}+2,\ldots,2^{n-1})\,, \tag{3.29}$$

where $[L_{n-1,i}, U_{n-1,i}]$ is the interval $I_{n-1,i}$ and $F_{n,i}$ is the value of the distribution function over $I_{n,i}$.

The induction hypothesis has been proved for $n=2$. Suppose that it holds for $n = 3,4,\ldots, k-1$. For $n=k$, the intervals $I_{k,i}(i=1,2,\ldots,2^{k-2})$ are generated from $I_{k-1,i}$ by the transformation B_1. Thus $x \in I_{k,i}$ for all $(x+1-\alpha_1)/\alpha_1 \in I_{k-1,i}$. Now let the value of the distribution function over $I_{k-1,i}$ be $F_{k-1,i}$. From equation (3.21) and the lower boundary condition (3.23) we get

$$F(x) = [\lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F_{k-1,i}-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{(x+1-\alpha_1)/\alpha_1} F(u)\,du,$$

$$(x \in I_{k,i})\,,$$

$$= [\lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F_{k-1,i}-\tfrac{1}{2}(\lambda_2-\lambda_1)\left[\int_{-1}^{L_{k-1,i}} + \int_{L_{k-1,i}}^{(x+1-\alpha_1)/\alpha_1} F_{k-1,i} \, du\right],$$

$$= [\lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F_{k-1,i}-\tfrac{1}{2}(\lambda_2-\lambda_1)\left[\int_{-1}^{L_{k-1,i}} F(u)\, du\right.$$

$$\left. + F_{k-1,i}[(x+1)/\alpha_1-1-L_{k-1,i})\right],$$

$$= [\lambda_1+\tfrac{1}{2}(\lambda_2-\lambda_1)(1+L_{k-1,i}]\, F_{k-1,i}-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{L_{k-1,i}} F(u)\, du\ , \quad (x\in I_{k,i}).$$

Since $F(x)$ is a constant over $I_{k,i}$, it may be denoted $F_{k,i}$. Thus

$$F_{k,i} = [\lambda_1+\tfrac{1}{2}(\lambda_2-\lambda_1)(1+L_{k-1,i})]F_{k-1,i}-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{L_{k-1,i}} F(u)\, du\ , \quad (x \in I_{k,i}).$$

This completes the proof of (3.28).

The same type of argument may be used to prove equation (3.29). In this case, the intervals $I_{k,i}$ $(i=2^{k-2}+1,\ 2^{k-2}+2,\ \ldots,2^{k-1})$ are generated from $I_{k-1,i-2^{k-2}}$ by the transformation B_2. Thus for $x \in I_{k,i}$, $(x-1+\alpha_2)/\alpha_2 \in I_{k-1,i-2^{k-2}}$ $(i > 2^{k-2})$. For simplicity of notation, let $i' = i-2^{k-2})$. From equation (3.21), the upper boundary condition (3.23) and the induction hypothesis we have

$$F(x) = \lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)+[1-\lambda_2+(1-x)(\lambda_2-\lambda_1)/(2\alpha_2)]F_{k-1,i'}$$

$$-\tfrac{1}{2}(\lambda_2-\lambda_1)\left[\int_{(x-1+\alpha_2)/\alpha_2}^{U_{k-1,i'}} F_{k-1,i'}\, du + \int_{U_{k-1,i'}}^{1} F(u)du + \int_{1}^{(x+1-\alpha_1)/\alpha_1} du\right],$$

$$= \lambda_1+(x+1)(\lambda_2-\lambda_1)/(2\alpha_1)+[1-\lambda_2+(1-x)(\lambda_2-\lambda_1)/(2\alpha_2)]F_{k-1,i'}$$

$$- \quad (\lambda_2-\lambda_1)\left[(U_{k-1,i'}-1-(x-1)/\alpha_2)F_{k-1,i'}+\int_{U_{k-1,i}}^{1} F(u)du + (x+1)/\alpha_1-2\right] .$$

Therefore

$$F(x)=\lambda_2+[1-\lambda_2-\tfrac{1}{2}(\lambda_2-\lambda_1)(U_{k-1,i'}-1)]F_{k-1,i'}-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{U_{k-1,i'}}^{1} F(u)\, du,$$

$$(x \in I_{k,i},\ i=2^{k-2}+1,\ldots,2^{k-1}).$$

Since $F(x)$ is constant over the intervals $I_{k,i}$ $(i > 2^{k-2})$ write $F(x)$ as $F_{k,i}$ so that

$$F_{k,i} = \lambda_2+[1-\lambda_2-\tfrac{1}{2}(\lambda_2-\lambda_1)(U_{k-1,i'}-1)]F_{k-1,i'}-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{U_{k-1,i'}}^{1} F(u)du .$$

Thus we have shown that the limiting distributions which satisfy (3.21), (3.23), (3.24) for $\alpha_1+\alpha_2 < 1$ are constant on a set of intervals $I_{n,i}$.

The same method of proof may be applied to the functional system (3.22) - (3.24) for $\alpha_1+\alpha_2 < 1$ to show that

$$F_{k,i} = [1-\lambda_1-\tfrac{1}{2}(\lambda_1-\lambda_2)(1-L_{k-1,i})]F_{k-1,i}+\tfrac{1}{2}(\lambda_1-\lambda_2)\int_{-1}^{L_{k-1,i}} F(u)\, du , \tag{3.30}$$

$$(x \in I_{k,i},\ i=1,2,\ldots,2^{k-2}) ,$$

and

$$F_{k,i} = 1-\lambda_1+[\lambda_1+\tfrac{1}{2}(\lambda_1-\lambda_2)(U_{k-1,i'}-1)]F_{k-1,i'}+\tfrac{1}{2}(\lambda_1-\lambda_2)\int_{U_{k-1,i'}}^{1} F(u)\, du \tag{3.31}$$

$$(x \in I_{k,i},\ i=2^{k-2}+1,\ 2^{k-2}+2,\ldots,2^{k-1}).$$

In theorem 2.1 it was shown that the set of intervals that are generated by the transformations B_1 and B_2 are disjoint and cover the domain $(-1,1)$ except for a non-denumerable set of points of Lebesgue measure zero.

In the case of $\alpha_1+\alpha_2 > 1$, almost nothing is known about the limiting distribution functions. The trial functions which satisfied the functional equation and boundary conditions in the Experimenter-Model do not satisfy the functional integral equations of this model.

CHAPTER IV

NUMERICAL SOLUTIONS FOR THE TRANSFORMED LIMITING DISTRIBUTIONS OF THE EXPERIMENTER-CONTROLLED MODEL.

§ 4.1 Introduction

Numerical approximations of the limiting distributions of response probabilities are of practical value for both applied statistics and pyschology. This may be especially true in the purely singular cases of $\alpha_1+\alpha_2 > 1$ where numerical tables may be the only method of characterizing the distribution. In this chapter a numerical method is described for tabulating the limiting distributions by a numerical solution of the functional system (2.26) - (2.28). This procedure is primarily intended for the cases $\alpha_1+\alpha_2 \geq 1$ since the distributions for $\alpha_1+\alpha_2 \leq 1$ can be uniquely determined by the algorithm of theorem 2.1. Unfortunately a general error analysis of the numerical procedure is impossible at this stage due to the lack of detailed information about the nature of the limiting distributions for $\alpha_1+\alpha_2 > 1$. Thus we have no general proof that the procedure will converge to the true distribution. The best that we can do is to compare the numerical results with the known properties of the distribution in the special cases where this can be done. An extremely favourable comparison is obtained between the numerical results and known absolutely continuous solutions, centro-symmetric distributions and a previously published Monte-Carlo approximation given in [1]. Thus we have a strong indication that the numerical procedure yields accurate solutions.

§ 4.2 Numerical Procedure

The major problem is to devise a fast and accurate method for tabulating the numerical values of the limiting distribution at a set of points in the domain (-1, 1). Since the functional system

$$(4.1) \qquad F(x) = \pi_1 F[(x+1-\alpha_1)/\alpha_1) + \pi_2 F[(x-1+\alpha_2)/\alpha_2) ,$$

$$(4.2) \qquad F(x) = 0 , \quad x \leq 1 \quad ; \quad F(x) = 1, \quad x \geq 1 \; ;$$

$$(4.3) \qquad 0 < \alpha_1, \; \alpha_2 < 1 \; ; \quad 0 \leq \pi_1 , \pi_2 \leq 1 \; ; \quad \pi_1 + \pi_2 = 1 ,$$

defines a unique distribution function (Theorem 2.3), we may reasonably expect that a numerical solution is possible.

To begin the numerical procedure we first define a set of tabular points $\{x_k\}$ $(k=0,1,2,\ldots,2N)$ lying in the closed region $[-1, 1]$. For simplicity, we will initially choose the tabular points to be equi-spaced such that the tabular interval $h = x_{k+1}-x_k=1/N$. On this tabular set $\{x_k\}$ the functional equation becomes

$$(4.4) \qquad F(kh-1) = \pi_1 F[kh-\alpha_1)/\alpha_1] + \pi_2 F[(kh-2+\alpha_2)/\alpha_2]. \quad (k=0,1,2,\ldots,2N) .$$

In general, the arguments $(kh-\alpha_1)/\alpha_1$ and $(kh-2+\alpha_2)/\alpha_2$ will not lie on any member of the chosen tabular set. Thus (4.4) defines a system of 2N-1 equations in 6N-3 unknown values of F. Imposing the boundary conditions (4.2) reduces the number of unknowns. The lower condition gives

$$(4.5) \qquad F(kh-1) = \pi_1 F[(kh-\alpha_1)/\alpha_1] , \qquad (k=1,2,3,\ldots,m) ;$$

where $\quad mh-1 \leq 1-2\alpha_2 \quad$ and $\quad (m+1)h-1 > 1-2\alpha_2$

The upper boundary condition implies

(4.6) $\quad F(kh-1) = \pi_1 + \pi_2 F[(kh-2+\alpha_2)/\alpha_2] \quad , \qquad (k=n,n+1,\ldots,2N-1) \; ;$

where $\quad nh-1 \geq 2\alpha_1-1 \quad$ and $\quad (n-1)h-1 < 2\alpha_1-1 \; .$

Because there are no more linearly independent conditions involving only the $6N-3-m-n$ values of F already given in (4.4) - (4.6) , the only feasible approach is to interpolate between the tabular values to approximate the off-tabular points. In this problem only the function values at the tabular points are at our disposal. This makes it rather inconvenient to apply the most accurate interpolation methods like the Bessel and Everett Interpolation formulas, since these procedures are expressed in terms of the differences of the function values. Furthermore the interpolating factor p $(x-ph = x_k)$ will probably be different for each point to be interpolated. This would necessitate calculating a whole set of Bessel or Everett coefficients for every off-tabular point.

Thus we try the simpler but less accurate Lagrange Interpolation formula.

Having chosen the interpolation formula, the next problem is to determine the order of interpolation that will give the best results. Because many of the solutions are known to be purely singular, a high order formula may introduce instability. Even the known absolutely continuous solutions have discontinuities in their higher derivatives. In the latter

case, a high order interpolation formula would either accentuate the effect of the discontinuities or smooth them over and hide them. For these reasons a four point Lagrange interpolation formula was tried, with the initial scheme to choose two tabular points on each side of the off-tabular argument.

After a number of numerical experiments, the following scheme proved to be far superior to any of the other variants tried. Given an off-tabular point x' lying between x_k and x_{k+1} $(k \neq 0,1, 2N-2, 2N-1)$ $F(x')$ was approximated by the formula

$$
\begin{aligned}
(4.7) \qquad F(x') = {} & \frac{(x'-x_k)(x'-x_{k+1})(x'-x_{k+2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k+1})(x_{k-1}-x_{k+2})} F(x_{k-1}) \\
& + \frac{(x'-x_{k-1})(x'-x_{k+1})(x'-x_{k+2})}{(x_k-x_{k-1})(x_k-x_{k+1})(x_k-x_{k+2})} F(x_k) \\
& + \frac{(x'-x_{k-1})(x'-x_k)(x'-x_{k+2})}{(x_{k+1}-x_{k-1})(x_{k+1}-x_k)(x_{k+1}-x_{k+2})} F(x_{k+2}) \\
& + \frac{(x'-x_{k-1})(x'-x_k)(x'-x_{k+1})}{(x_{k+2}-x_{k-1})(x_{k+2}-x_k)(x_{k+2}-x_{k+1})} F(x_{k+2}) .
\end{aligned}
$$

For $-1 < x' < x_2$ the interpolation formula was changed to:

$$
\begin{aligned}
(4.8) \qquad F(x') = {} & \frac{(x'-x_2)(x'-x_3)(x'-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} F(x_1) + \frac{(x'-x_1)(x'-x_3)(x'-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} F(x_2) \\
& + \frac{(x'-x_1)(x'-x_2)(x'-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} F(x_3) + \frac{(x'-x_1)(x'-x_2)(x'-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} F(x_4) .
\end{aligned}
$$

And for $x_{2N-2} < x' < 1$, the formula used was:

$$(4.9) \qquad F(x') = \frac{(x'-x_{2N-2})(x'-x_{2N-1})(x'-x_{2N})}{(x_{2N-3}-x_{2N-2})(x_{2N-3}-x_{2N-1})(x_{2N-3}-x_{2N})} F(x_{2N-3})$$

$$+ \frac{(x'-x_{2N-3})(x'-x_{2N-1})(x'-x_{2N})}{(x_{2N-2}-x_{2N-3})(x_{2N-2}-x_{2N-1})(x_{2N-2}-x_{2N})} F(x_{2N-2})$$

$$+ \frac{(x'-x_{2N-3})(x'-x_{2N-2})(x'-x_{2N})}{(x_{2N-1}-x_{2N-3})(x_{2N-1}-x_{2N-2})(x_{2N-1}-x_{2N})} F(x_{2N-1})$$

$$+ \frac{(x'-x_{2N-3})(x'-x_{2N-2})(x'-x_{2N-1})}{(x_{2N}-x_{2N-3})(x_{2N}-x_{2N-2})(x_{2N}-x_{2N-1})} F(x_{2N}) \ .$$

To set up the system of linear equations in $F(x_k)(k=1,2,\ldots 2N-1)$, the form of (4.4)-(4.6) was changed to

$$-F(kh-1) + \pi_1 F[(kh-\alpha_1)/\alpha_1] = 0 \qquad (k=1,2,\ldots,m) \ ;$$

$$(4.10) \qquad -F(kh-1)+\pi_1 F[(kh-\alpha_1)/\alpha_1]+\pi_2 F[(kh-2+\alpha_2)/\alpha_2] = 0 \qquad (k=m+1,m+2,\ldots,n-1);$$

$$-F(kh-1) + \pi_2 F[(kh-2+\alpha_2)/\alpha_2] = -\pi_1 \ ; \qquad (k=n,n+1,\ldots,2N-1) \ .$$

The final system of equations has the following form.

$$
\begin{bmatrix}
-1+C_{1,1}\pi_2 & C_{1,2}\pi_1 & C_{1,3}\pi_1 & C_{1,4}\pi_1 & 0 & 0 & 0 \\
0 & -1+C_{2,1}\pi_1 & C_{2,2}\pi_1 & C_{2,3}\pi_1 & C_{2,4}\pi_1 & 0 & 0 \\
\pi_2 C'_{3,2} & \pi_2 C'_{3,2} & \pi_2 C'_{3,3}-1+\pi_1 C_{3,1} & \pi_2 C'_{3,4}+\pi_1 C_{3,2} & \pi_1 C_{3,4} & \pi_1 C_{3,4} & 0 \\
\pi_2 C'_{4,1} & \pi_2 C'_{4,2} & \pi_2 C'_{4,3} & \pi_2 C'_{4,4}-1 & \pi_1 & 0 & 0 \\
0 & \pi_2 C'_{5,1} & \pi_2 C'_{5,2} & \pi_2 C'_{5,3} & \pi_2 C'_{5,4}-1+\pi_1 C_{5,1} & \pi_1 C_{5,2} & \pi_1 C_{5,3} \\
0 & 0 & 0 & \pi_2 C'_{6,1} & \pi_2 C'_{6,2}+\pi_1 C_{6,1} & \pi_2 C'_{6,3}-1+\pi_1 C_{6,2} & \pi_2 C'_{6,4}+\pi_1 C_{6,3} \\
0 & 0 & 0 & 0 & \pi_2 C'_{7,1} & \pi_2 C'_{7,2} & \pi_2 C'_{7,3}-1
\end{bmatrix}
\begin{bmatrix}
F(x_1) \\ F(x_2) \\ F(x_3) \\ F(x_4) \\ F(x_5) \\ F(x_6) \\ F(x_7)
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ -\pi_1 C_{5,4} \\ -\pi_1 C_{6,4} \\ -\pi_2 C'_{7,4}-\pi_1
\end{bmatrix}
$$

Figure 5: Form of the system of linear equations for $F(x_i)$ $(i=1,\ldots,2N-1)$

where c_{ij} are the Lagrangian Interpolation coefficients for the argument

$$(kh - \alpha_1)/\alpha_1 \quad ,$$

and c'_{ij} are the Lagrangian Interpolation coefficient for the argument

$$(kh-2+\alpha_2)/\alpha_2 \quad .$$

A Fortran program was written to set up and solve the above system of equations given any set of α_1, α_2, π_1, and π_2, along with a tabular interval h. The final matrix equation was solved using a Gaussian-elimination procedure which chooses the best pivots of each column.

The exact procedure used to set up the system of equations for tabulating $F(x_k)$ $(k=1,2,\ldots,2N-1)$ is best explained by the following ALGOL program.

```
procedure Experimenter Limiting Distributions;
    real array A[1: M, 1: M+1], x[0: M+1], F[0: M], c[1:4] ;
    real h, alfa 1, alfa 2, pi 1, arg 1, arg 2, swtch, pi 2, pi ;
    Boolean leap, jump; integer i, j, k, max, lead;

comment This procedure computes the values of the transformed
limiting distribution function of the Experimenter-Controlled Model
at a set of points x[k] in the domain (-1,1). These values are
determined by a system of linear equations in the unknowns F(x[k]) ;

procedure Lagrange (lead, max, arg, x, c) ;
    real d ;              integer i, j, k, ℓ ;
    value lead, max, arg ;
```

<u>comment</u> Lagrange is a sub-procedure which calculates the Lagrange Interpolation coefficients c[i] given the value of the abscissa 'arg', the set of tabular points x[i] , and the index 'lead' which defines the set of points to be used in the interpolation. The value 'max' determines if the upper boundary is used in the interpolation.

```
begin
    for  i: = 1  step 1  until 4 do
    begin
        d: = 1.0  ;  k: = lead + i ;
        for  j: = 1  step  1  until  4  do
        begin
            if  i = j  then go to cycle else
            begin
                ℓ: = lead + j ;
                d: = d × (arg - x[ℓ])/(x[k]-x[ℓ] ;
            end
    cycle: end  loop with index  j ;
            c[i] : = d ;
        end  main loop for calculating  c[i] ;
    if  lead + 3 = max  then   c[4] : = - c[4] ;
end  Lagrange   This completes the sub-procedure Lagrange;
```

```
begin Comment The following program segment is the main body of the procedure
    which sets up the matrix equation;
   start: read (h, alfa 1, alfa 2, pi 1) ;
          max: = 2.0/h ; pi 2: = 1.0 - pi 1 ;
          for i: = 0 step 1 until max + 1 do x[i]: = h × i - 1.0 ;
          arg 1: = 1.0 - alfa 1; arg 2: = alfa 2 - 1.0; swtch: = 1.0 -
          alfa 1 - alfa 2 ; leap: = true ;
          for i: = 1 step 1 until max -1 do
          begin comment This loop determines the i^th row of the matrix;
             for j: = 1 step 1 until max do A[i, j]: = 0.0 ;
             A[i,i]: = -1.0 ; pi: = pi 1 ; jump: = true ;
             if leap them go to execute else go to skip ;
   execute: arg: = (x[i] + arg 1)/alfa 1 ;
            if arg ≥ 1.0 then go to set else
            begin
   search : for j: = 1, j+1 while x[j] < arg do ;
            if x[j] = arg then A[i,j]: = pi + A[i,j]
            else if j< 2 then lead: = 0
            else if j > max - 2 then lead: = max-4
            else lead: = j - 3 ;
            end
            Lagrange (lead, max, arg, x, c) ;
            for k: = 1 + lead step 1 until 4 + lead do
            A[i,k]: = A[i,k] + pi × c[k-lead] ;
```

```
            go to reset;
set:        leap: = false;
skip:       A[i, max]: = -pi ;
reset:      if jump then begin if x[i] < swtch then go to restart end
            else begin comment this part re-initializes to interpolate
                          for the smallest argument;
                    pi: = pi 2 ;  arg: = (x[i] + arg 2)/alfa 2 ;
                    jump: = false;
                    go to search;
                 end

restart:  end This ends the main loop for setting up the matrix equation;

    GAUSEL (max -1, A, F) ;
    Comment GAUSEL is a Gaussian elimination procedure for solving sets
            of linear equations. Given matrix A of order (max -1) × (max),
            this procedure solves the set of equations by Gaussian Elimination
            with best pivots and leaves the solution in the column vector F;
    F[0]: = 0.0 ; F(max): = 1.00;
    Punch (alfa 1, pi 1, alfa 2);
    Punch (x[i], F[i], i: = 0, max);
    Comment The above segment merely punches out the values of the parameters
            then a table of x[i] and F[i];
    go to start;
end Experimenter Limiting Distributions;
```

§ 4.3 Accuracy of the Numerical Results.

The Fortran program for executing the algorithm of §4.2 was run on the I.B.M. 1620 system of the University of Alberta. The calculations were done using hardware floating point carrying an eight digit mantissa and two digit exponent. A number of test cases were calculated to compare the numerical answers with values which could be determined from theoretical considerations. All the test cases indicated a very satisfactory accuracy-more than adequate for any practical problem. However, because of the nature of this problem, no error bounds on the numerical process itself have been determined.

The simplest of the test cases depended upon the property of centro-symmetry of the transformed limiting distributions for $\alpha_1 = \alpha_2 = \alpha$, $\pi_1 = \pi_2 = \frac{1}{2}$. This property may be proved from a result of McGregor and Hui ([2]p.1282). These authors showed that in the equal alpha case, the characteristic function of the transformed limiting distribution reduced to

$$\varphi(\theta) = \prod_{n=o}^{\infty} [\pi_1 e^{-i\theta(1-\alpha_1)\alpha_1^n} + \pi_2 e^{i\theta(1-\alpha)\alpha^n}] . \tag{4.11}$$

Choosing $\pi_1 = \pi_2 = \frac{1}{2}$, the above characteristic function may be written as an infinite product of cosines. Thus the frequency function must be symmetric about zero and the distribution function centro-symmetric about $(0,\frac{1}{2})$. The first test consisted of tabulating the distribution functions corresponding to $\pi_1 = \pi_2 = \frac{1}{2}$, with α taking the values .55(05) .95. The numerical solutions for the tabular interval $h=0.1$ are displayed in table 2 (pg.92). In every case $F(x) + F(-x) = 1+\epsilon$ where $|\epsilon| < 5 \times 10^{-5}$. The

apparently bad solutions for $\alpha = 0.9$ and 0.95 do not indicate a failing of the procedure, but merely reflect the fact that as $\alpha \to 1$, the arguments $(x+1-\alpha_1)/\alpha_1$ and $(x-1+\alpha_2)/\alpha_2$ tend to x so that the functional equations (4.1) become a tautology. By halving the tabular interval so that $h = 0.05$, the resulting solutions appear to be accurate to six or seven figures (cf. pages 101 and 102).

Another check on the procedure is the comparison of the numerical solution by a system of linear equations with a Monte-Carlo solution given by Bush and Mosteller in [1] p.133. These authors calculated on approximate limiting distribution by a 1000 - trial stat-rat for the parameter values:

$$\pi_1 = \pi_2 = 0.50 \; ;$$
$$\alpha_1 = 0.60 \; , \quad \alpha_2 = 0.90 \; ;$$
$$\lambda_1 = 0.75 \; , \quad \lambda_2 = 0.10 \; .$$

The results of their calculation are displayed graphically in figure 6. Since the numerical solution by the method of linear equations tabulates the transformed distribution, an inverse transformation was made to map the limiting distribution onto the domain [.10, .75]. The results of this solution are displayed graphically in figure 7 .

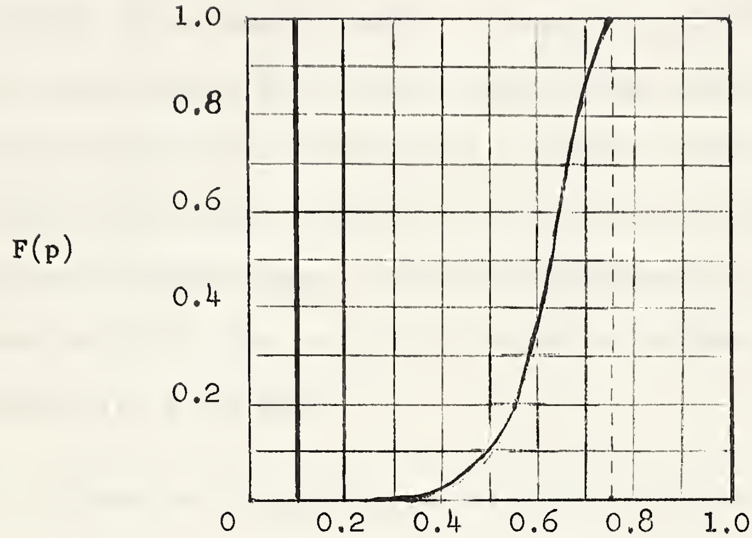

Figure 6: The approximate asymptotic cumulative distribution of p obtained from a 1000-trial stat-rat for Experimenter-Controlled events $Q_1 p = 0.3 + 0.6\, p$, $Q_2 p = 0.01 + 0.9\, p$, $\pi_1 = \pi_2 = 0.5$. The vertical dotted lines show the trapping limits $\lambda_2 = 0.10$ and $\lambda_1 = 0.75$. (Reproduced from [1] p. 133).

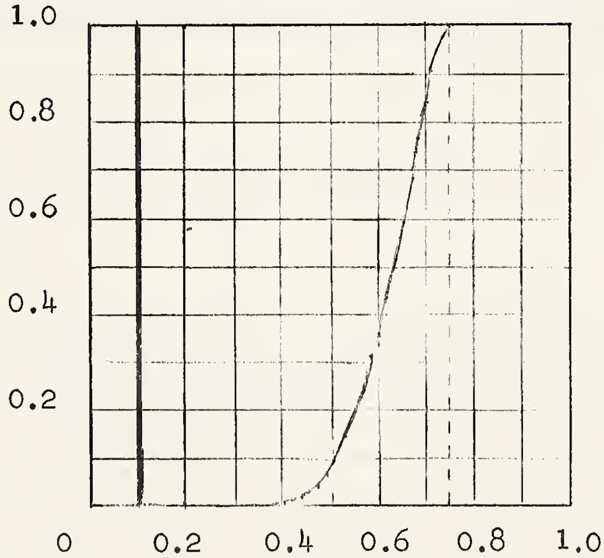

Figure 7: The approximate limiting distribution of p obtained from a system of 39 linear equations in $F(p)$ for

$$\pi_1=\pi_2=0.5\ ,\quad \alpha_1=0.60\ ,\quad \alpha_2 = 0.90,\quad \lambda_1 = 0.75\ ,\quad \lambda_2=0.10\ .$$

A second set of tables was calculated for values of the parameters which yield known absolutely continuous distributions (Zidek [9]). These tables include the parameter values $\pi_1=\pi_2=\frac{1}{2}$, $\alpha_1=\alpha_2=\frac{1}{2}^{1/n}$, (n=2,3,4,...,10). A number of spot checks were made by calculating numerical values of the distribution from the known absolutely continuous functions and comparing these results with values taken from the numerical tables. Table 1 displays some of these checks. Since the functions are centro-symmetric, only values of $F(x)$ for $x \in (-1, 0)$ need be included. Note that the maximum error is $\leq \pm\ 2\times10^{-4}$.

A final set of tables was calculated for some representative values of the parameters to display the behaviour of the distributions for $\alpha_1+\alpha_2 \geq 1$.

Although it has not been proved that the numerical procedure converges to the true solution, the above tests indicate that a reasonable amount of confidence may be placed on the numerical solution.

n	x	True Value of F(x)	Approximate Value of F(x)
2	-.900	.005 151	.005 151
	-.750	.032 197	.032 197
	-.600	.082 430	.082 430
	-.450	.155 835	.155 850
	-.300	.252 426	.252 431
	-.150	.371 960	.371 957
3	-.900	.000 593	.000 593
	-.800	.004 746	.004 746
	-.400	.127 834	.127 828
	-.200	.288 896	.288 868
	-.100	.390 917	.390 880
4	-.900	.000 072	.000 069
	-.700	.005 817	.005 817
	-.500	.044 871	.044 740
	-.400	.090 959	.091 001
	-.200	.256 953	.256 741
6	-.900	.000 011	.000 000
	-.800	.000 073	.000 068
	-.700	.000 829	.000 824

Table 1. Comparison of Values of the distribution function calculated from the absolutely continuous functions and those from the numerical tables for $\pi_1=\pi_2=\frac{1}{2}$, $\alpha_1=\alpha_2=(\frac{1}{2})^{1/n}$.

ALPHA1= .55000000 PI1= .500 ALPHA2= .55000000

-1.000000	0.000000
-.900000	.034784
-.800000	.078173
-.700000	.123252
-.600000	.170079
-.500000	.214064
-.400000	.285946
-.300000	.329920
-.200000	.376747
-.100000	.421827
0.000000	.500000
.100000	.578173
.200000	.623252
.300000	.670079
.400000	.714064
.500000	.785946
.600000	.829920
.700000	.876747
.800000	.921827
.900000	.965216
1.000000	1.000000

TABLE 2. CENTRO-SYMMETRIC LIMITING DISTRIBUTIONS OF THE EXPERIMENTER -CONTROLLED MODEL

ALPHA1= .60000000 PI1= .500 ALPHA2= .60000000

X	F
-1.000000	0.000000
-.900000	.022137
-.800000	.055585
-.700000	.094174
-.600000	.146191
-.500000	.188348
-.400000	.250011
-.300000	.311753
-.200000	.353847
-.100000	.428014
0.000000	.500022
.100000	.572483
.200000	.646190
.300000	.688348
.400000	.750011
.500000	.811753
.600000	.853847
.700000	.905876
.800000	.944437
.900000	.978309
1.000000	1.000000

TABLE 2.

ALPHA1= .65000000   PI1= .500   ALPHA2= .65000000

X	F
-1.000000	0.000000
-.900000	.012314
-.800000	.037454
-.700000	.071894
-.600000	.114535
-.500000	.163448
-.400000	.220274
-.300000	.279730
-.200000	.348929
-.100000	.422917
0.000000	.500020
.100000	.577091
.200000	.651236
.300000	.720274
.400000	.779730
.500000	.836615
.600000	.885462
.700000	.928126
.800000	.962555
.900000	.987787
1.000000	1.000000

TABLE 2.

ALPHA1= .70000000 PI1= .500 ALPHA2= .70000000

X	F
-1.000000	0.000000
-.900000	.005840
-.800000	.022445
-.700000	.049359
-.600000	.086376
-.500000	.133107
-.400000	.189968
-.300000	.255840
-.200000	.332476
-.100000	.416256
0.000000	.499999
.100000	.583748
.200000	.667515
.300000	.744168
.400000	.810030
.500000	.866896
.600000	.913623
.700000	.950640
.800000	.977547
.900000	.994168
1.000000	1.000000

TABLE 2.

ALPHA1= .75000000   PI1= .500   ALPHA2= .75000000

X	F
-1.000000	0.000000
-.900000	.002098
-.800000	.011213
-.700000	.029771
-.600000	.059542
-.500000	.101911
-.400000	.158148
-.300000	.228889
-.200000	.312100
-.100000	.403494
0.000000	.500011
.100000	.596507
.200000	.687904
.300000	.771152
.400000	.841832
.500000	.898099
.600000	.940457
.700000	.970228
.800000	.988823
.900000	.997880
1.000000	1.000000

TABLE 2.

ALPHA1= .80000000 PI1= .500 ALPHA2= .80000000

X	F
-1.000000	0.000000
-.900000	.000477
-.800000	.004073
-.700000	.014365
-.600000	.035111
-.500000	.070222
-.400000	.123195
-.300000	.195321
-.200000	.285110
-.100000	.388785
0.000000	.499999
.100000	.611205
.200000	.714888
.300000	.804674
.400000	.876791
.500000	.929777
.600000	.964888
.700000	.985634
.800000	.995913
.900000	.999528
1.000000	1.000000

TABLE 2.

ALPHA1= .85000000 PI1= .500 ALPHA2= .85000000

X	F
-1.000000	0.000000
-.900000	.000078
-.800000	.000753
-.700000	.004466
-.600000	.015300
-.500000	.039277
-.400000	.083464
-.300000	.153030
-.200000	.249328
-.100000	.368165
0.000000	.499975
.100000	.631772
.200000	.750625
.300000	.846901
.400000	.916512
.500000	.960607
.600000	.984703
.700000	.995544
.800000	.999148
.900000	.999969
1.000000	1.000000

TABLE 2.

ALPHA1= .90000000 PI1= .500 ALPHA2= .90000000

X	F
-1.000000	0.000000
-.900000	.000027
-.800000	-.000007
-.700000	.000400
-.600000	.002950
-.500000	.013037
-.400000	.040532
-.300000	.098470
-.200000	.196999
-.100000	.335796
0.000000	.499981
.100000	.664165
.200000	.802962
.300000	.901494
.400000	.959420
.500000	.986945
.600000	.996974
.700000	.999628
.800000	.999959
.900000	1.000002
1.000000	1.000000

TABLE 2.

ALPHA1= .95000000 PI1= .500 ALPHA2= .95000000

X	F
-1.000000	0.000000
-.900000	-.000002
-.800000	-.000002
-.700000	-.000030
-.600000	-.000259
-.500000	-.000430
-.400000	.004168
-.300000	.031181
-.200000	.113058
-.100000	.275054
0.000000	.500000
.100000	.724946
.200000	.886942
.300000	.968819
.400000	.995832
.500000	1.000431
.600000	1.000260
.700000	1.000032
.800000	1.000001
.900000	1.000000
1.000000	1.000000

TABLE 2.

ALPHA1= .90000000   PI1= .500   ALPHA2= .90000000

X	F
-1.000000	0.000000
-.950000	0.000000
-.900000	0.000000
-.850000	.000003
-.800000	.000030
-.750000	.000139
-.700000	.000470
-.650000	.001305
-.600000	.003136
-.550000	.006734
-.500000	.013191
-.450000	.023917
-.400000	.040552
-.350000	.064817
-.300000	.098277
-.250000	.142082
-.200000	.196687
-.150000	.261653
-.100000	.335556
-.050000	.416035
0.000000	.499999
.050000	.583964
.100000	.664443
.150000	.738346
.200000	.803312
.250000	.857917
.300000	.901722
.350000	.935182
.400000	.959447
.450000	.976082
.500000	.986808
.550000	.993265
.600000	.996863
.650000	.998694
.700000	.999529
.750000	.999862
.800000	.999970
.850000	.999996
.900000	1.000000
.950000	1.000000
1.000000	1.000000

TABLE 2.

ALPHA1= .95000000 PI1= .500 ALPHA2= .95000000

X	F
-1.000000	0.000000
-.950000	0.000000
-.900000	0.000000
-.850000	0.000000
-.800000	0.000000
-.750000	0.000000
-.700000	0.000000
-.650000	0.000000
-.600000	.000019
-.550000	.000125
-.500000	.000547
-.450000	.001883
-.400000	.005436
-.350000	.013594
-.300000	.030062
-.250000	.059641
-.200000	.107338
-.150000	.176811
-.100000	.268617
-.050000	.379006
0.000000	.500000
.050000	.620993
.100000	.731382
.150000	.823187
.200000	.892660
.250000	.940357
.300000	.969937
.350000	.986404
.400000	.994563
.450000	.998116
.500000	.999452
.550000	.999874
.600000	.999980
.650000	.999999
.700000	1.000000
.750000	1.000000
.800000	1.000000
.850000	1.000000
.900000	1.000000
.950000	1.000000
1.000000	1.000000

TABLE 2.

Table 3 below displays the numerical solutions of the absolutely continuous distribution functions corresponding to $\pi_1=\pi_2=\frac{1}{2}$, $\alpha_1=\alpha_2=(\frac{1}{2})^{1/n}$, $(n=2,3,\ldots,10)$. The following tables have been rounded to four significant digits.

ALPHA1= .70710678 PI1= .500 ALPHA2= .70710678

X	F
-1.000	0.0000
-.950	.0013
-.900	.0052
-.850	.0116
-.800	.0206
-.750	.0322
-.700	.0464
-.650	.0631
-.600	.0824
-.550	.1043
-.500	.1288
-.450	.1559
-.400	.1855
-.350	.2177
-.300	.2524
-.250	.2898
-.200	.3297
-.150	.3720
-.100	.4146
-.050	.4573
0.000	.5000
.050	.5427
.100	.5854
.150	.6280
.200	.6703
.250	.7102
.300	.7476
.350	.7823
.400	.8145
.450	.8441
.500	.8712
.550	.8957
.600	.9176
.650	.9369
.700	.9536
.750	.9678
.800	.9794
.850	.9884
.900	.9948
.950	.9987
1.000	1.0000

TABLE 3. NUMERICAL SOLUTION OF THE ABSOLUTELY CONTINUOUS LIMITING DISTRIBUTIONS FOR N=2-10.

ALPHA1= .79370053   PI1= .500   ALPHA2= .79370053

X	F
-1.000	0.0000
-.950	.0001
-.900	.0006
-.850	.0020
-.800	.0047
-.750	.0093
-.700	.0160
-.650	.0254
-.600	.0380
-.550	.0541
-.500	.0742
-.450	.0987
-.400	.1278
-.350	.1616
-.300	.1999
-.250	.2425
-.200	.2889
-.150	.3385
-.100	.3909
-.050	.4450
0.000	.5000
.050	.5550
.100	.6091
.150	.6615
.200	.7111
.250	.7575
.300	.8001
.350	.8384
.400	.8722
.450	.9013
.500	.9258
.550	.9459
.600	.9620
.650	.9746
.700	.9840
.750	.9907
.800	.9953
.850	.9980
.900	.9994
.950	.9999
1.000	1.0000

TABLE 3.

ALPHA1= .84089642   PI1= .500   ALPHA2= .84089642

X	F
-1.000	0.0000
-.950	0.0000
-.900	.0001
-.850	.0004
-.800	.0011
-.750	.0028
-.700	.0058
-.650	.0108
-.600	.0184
-.550	.0294
-.500	.0447
-.450	.0650
-.400	.0910
-.350	.1231
-.300	.1615
-.250	.2062
-.200	.2567
-.150	.3125
-.100	.3725
-.050	.4355
0.000	.5000
.050	.5645
.100	.6275
.150	.6875
.200	.7433
.250	.7938
.300	.8385
.350	.8769
.400	.9090
.450	.9350
.500	.9553
.550	.9706
.600	.9816
.650	.9892
.700	.9942
.750	.9972
.800	.9989
.850	.9996
.900	.9999
.950	1.0000
1.000	1.0000

TABLE 3.

ALPHA1= .87055056 PI1= .500 ALPHA2= .87055056

X	F
-1.000	0.0000
-.950	0.0000
-.900	0.0000
-.850	.0001
-.800	.0003
-.750	.0009
-.700	.0022
-.650	.0047
-.600	.0092
-.550	.0164
-.500	.0276
-.450	.0437
-.400	.0659
-.350	.0952
-.300	.1322
-.250	.1774
-.200	.2304
-.150	.2906
-.100	.3568
-.050	.4273
0.000	.5000
.050	.5727
.100	.6432
.150	.7094
.200	.7696
.250	.8226
.300	.8678
.350	.9048
.400	.9341
.450	.9563
.500	.9724
.550	.9836
.600	.9908
.650	.9953
.700	.9978
.750	.9991
.800	.9997
.850	.9999
.900	1.0000
.950	1.0000
1.000	1.0000

TABLE 3.

ALPHA1= .89089872 PI1= .500 ALPHA2= .89089872

X	F
-1.000	0.0000
-.950	0.0000
-.900	0.0000
-.850	0.0000
-.800	.0001
-.750	.0003
-.700	.0008
-.650	.0021
-.600	.0046
-.550	.0093
-.500	.0172
-.450	.0297
-.400	.0483
-.350	.0744
-.300	.1093
-.250	.1538
-.200	.2081
-.150	.2716
-.100	.3429
-.050	.4200
0.000	.5000
.050	.5800
.100	.6571
.150	.7284
.200	.7919
.250	.8462
.300	.8907
.350	.9256
.400	.9517
.450	.9703
.500	.9828
.550	.9907
.600	.9954
.650	.9979
.700	.9992
.750	.9997
.800	.9999
.850	1.0000
.900	1.0000
.950	1.0000
1.000	1.0000

TABLE 3.

ALPHA1= .90572366 PI1= .500 ALPHA2= .90572366

X	F
-1.000	0.0000
-.950	0.0000
-.900	0.0000
-.850	0.0000
-.800	0.0000
-.750	.0001
-.700	.0003
-.650	.0009
-.600	.0024
-.550	.0053
-.500	.0109
-.450	.0204
-.400	.0357
-.350	.0587
-.300	.0910
-.250	.1342
-.200	.1889
-.150	.2548
-.100	.3304
-.050	.4133
0.000	.5000
.050	.5867
.100	.6696
.150	.7452
.200	.8111
.250	.8658
.300	.9090
.350	.9413
.400	.9643
.450	.9796
.500	.9891
.550	.9947
.600	.9976
.650	.9991
.700	.9997
.750	.9999
.800	1.0000
.850	1.0000
.900	1.0000
.950	1.0000
1.000	1.0000

TABLE 3.

ALPHA1= .91700404   PI1= .500   ALPHA2= .91700404

X	F
-1.000	0.0000
-.950	0.0000
-.900	0.0000
-.850	0.0000
-.800	0.0000
-.750	0.0000
-.700	.0001
-.650	.0004
-.600	.0012
-.550	.0031
-.500	.0069
-.450	.0141
-.400	.0266
-.350	.0465
-.300	.0762
-.250	.1176
-.200	.1721
-.150	.2397
-.100	.3190
-.050	.4071
0.000	.5000
.050	.5929
.100	.6810
.150	.7603
.200	.8279
.250	.8824
.300	.9238
.350	.9535
.400	.9734
.450	.9859
.500	.9931
.550	.9969
.600	.9988
.650	.9996
.700	.9999
.750	1.0000
.800	1.0000
.850	1.0000
.900	1.0000
.950	1.0000
1.000	1.0000

TABLE 3.

ALPHA1= .92587471 PI1= .500 ALPHA2= .92587471

X	F
-1.000	0.0000
-.950	0.0000
-.900	0.0000
-.850	0.0000
-.800	0.0000
-.750	0.0000
-.700	0.0000
-.650	.0002
-.600	.0006
-.550	.0018
-.500	.0044
-.450	.0098
-.400	.0199
-.350	.0370
-.300	.0640
-.250	.1034
-.200	.1573
-.150	.2260
-.100	.3084
-.050	.4014
0.000	.5000
.050	.5986
.100	.6916
.150	.7740
.200	.8427
.250	.8966
.300	.9360
.350	.9630
.400	.9801
.450	.9902
.500	.9956
.550	.9982
.600	.9994
.650	.9998
.700	1.0000
.750	1.0000
.800	1.0000
.850	1.0000
.900	1.0000
.950	1.0000
1.000	1.0000

TABLE 3.

ALPHA1= .93303299   PI1= .500   ALPHA2= .93303299

X	F
-1.000	0.0000
-.950	0.0000
-.900	0.0000
-.850	0.0000
-.800	0.0000
-.750	0.0000
-.700	0.0000
-.650	.0001
-.600	.0003
-.550	.0010
-.500	.0028
-.450	.0068
-.400	.0149
-.350	.0296
-.300	.0539
-.250	.0913
-.200	.1441
-.150	.2135
-.100	.2986
-.050	.3960
0.000	.5000
.050	.6040
.100	.7014
.150	.7865
.200	.8559
.250	.9087
.300	.9461
.350	.9704
.400	.9851
.450	.9932
.500	.9972
.550	.9990
.600	.9997
.650	.9999
.700	1.0000
.750	1.0000
.800	1.0000
.850	1.0000
.900	1.0000
.950	1.0000
1.000	1.0000

TABLE 3.

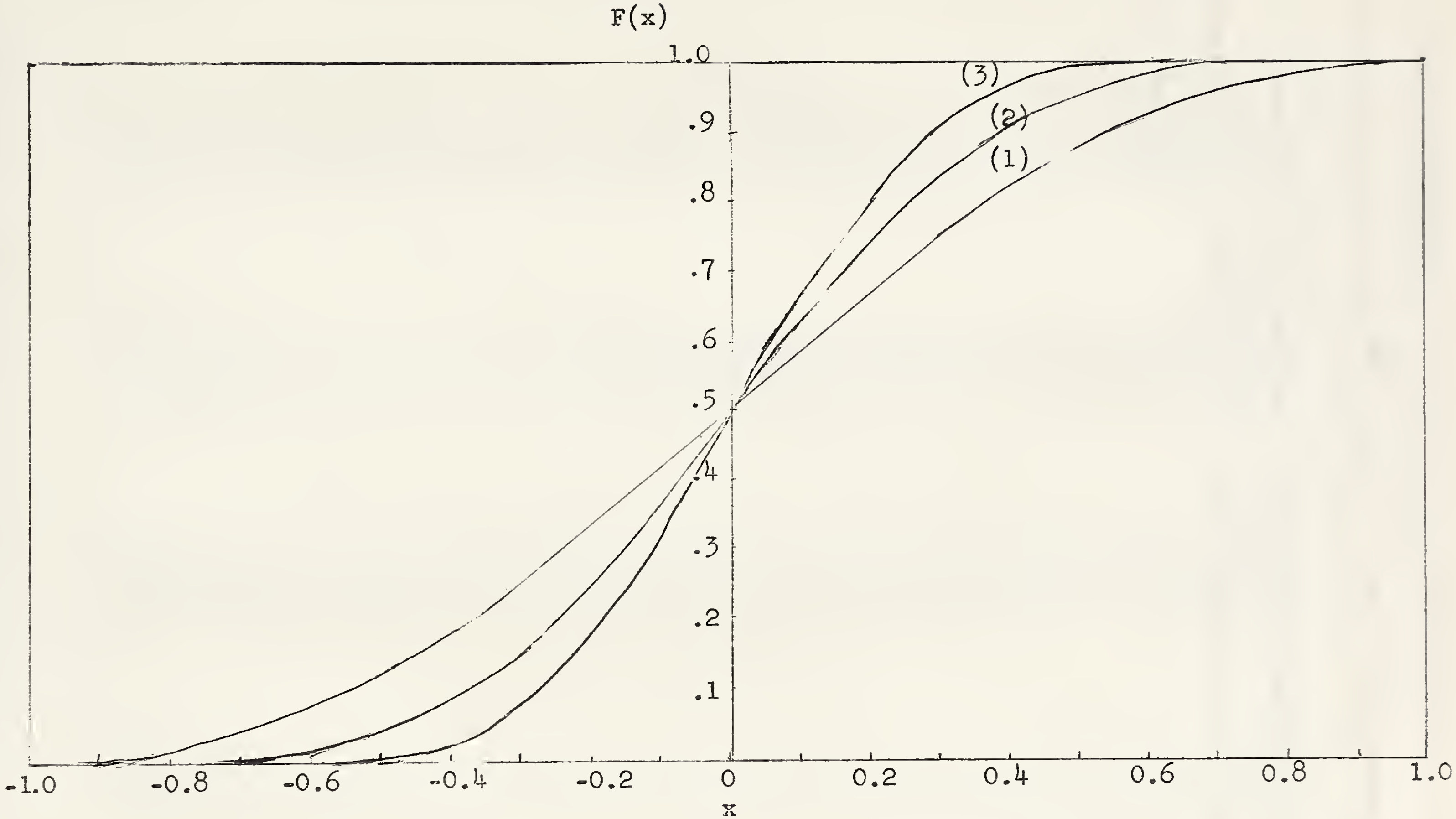

Figure 8: Absolutely continuous limiting distributions for $\alpha_1=\alpha_2=(\frac{1}{2})^{1/2}$, $(\frac{1}{2})^{1/4}$, $(\frac{1}{2})^{1/7}$.

(1) $\alpha = (\frac{1}{2})^{1/2}$
(2) $\alpha = (\frac{1}{2})^{1/4}$
(3) $\alpha = (\frac{1}{2})^{1/7}$

The following table displays the behaviour of the transformed limiting distribution functions for selected values of the parameters.

Figure 8 page 113 contains a graph of the absolutely continuous distributions for $\alpha = (\frac{1}{2})^{1/2}$, $(\frac{1}{2})^{1/4}$, $(\frac{1}{2})^{1/7}$ (see also the corresponding results of Zidek [9]).

ALPHA1= .80000000 ALPHA2= .20000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	0.0000	.0002	.0250
-.900	0.0000	0.0000	.0009	.0500
-.850	0.0000	0.0000	.0024	.0750
-.800	0.0000	.0001	.0046	.1000
-.750	0.0000	.0001	.0077	.1250
-.700	0.0000	.0003	.0117	.1500
-.650	0.0000	.0006	.0166	.1750
-.600	0.0000	.0010	.0226	.2000
-.550	0.0000	.0017	.0295	.2250
-.500	0.0000	.0025	.0377	.2500
-.450	.0001	.0041	.0476	.2750
-.400	.0001	.0045	.0547	.3000
-.350	.0002	.0081	.0714	.3250
-.300	.0003	.0107	.0816	.3500
-.250	.0003	.0113	.0911	.3750
-.200	.0007	.0178	.1134	.4000
-.150	.0017	.0268	.1332	.4250
-.100	.0016	.0261	.1387	.4500
-.050	.0018	.0299	.1579	.4750
0.000	.0037	.0446	.1890	.5000
.050	.0085	.0664	.2202	.5250
.100	.0080	.0642	.2235	.5500
.150	.0081	.0681	.2422	.5750
.200	.0096	.0789	.2717	.6000
.250	.0186	.1116	.3150	.6250
.300	.0418	.1640	.3643	.6500
.350	.0400	.1603	.3667	.6750
.400	.0401	.1627	.3819	.7000
.450	.0410	.1741	.4124	.7250
.500	.0481	.1972	.4528	.7500
.550	.0595	.2389	.4989	.7750
.600	.2000	.4000	.6000	.8000
.650	.2000	.4001	.6031	.8250
.700	.2000	.4015	.6151	.8500
.750	.2003	.4068	.6365	.8750
.800	.2030	.4268	.6756	.9000
.850	.2149	.4669	.7260	.9250
.900	.2385	.5183	.7811	.9500
.950	.3602	.6441	.8546	.9750
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4. NUMERICAL APPROXIMATION OF SOME LIMITING DISTRIBUTIONS OF THE EXPERIMENTER-CONTROLLED MODEL FOR SELECTED VALUES OF THE PARAMETERS

ALPHA1= .80000000 ALPHA2= .40000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	0.0000	.0002	.0295
-.900	0.0000	0.0000	.0012	.0591
-.850	0.0000	0.0000	.0030	.0886
-.800	0.0000	.0001	.0058	.1181
-.750	0.0000	.0002	.0097	.1477
-.700	0.0000	.0004	.0148	.1771
-.650	0.0000	.0007	.0209	.2070
-.600	0.0000	.0013	.0286	.2359
-.550	0.0000	.0021	.0371	.2663
-.500	0.0000	.0032	.0477	.2949
-.450	.0001	.0048	.0589	.3262
-.400	.0001	.0065	.0715	.3528
-.350	.0002	.0099	.0876	.3848
-.300	.0004	.0128	.1019	.4150
-.250	.0005	.0162	.1192	.4410
-.200	.0010	.0229	.1411	.4732
-.150	.0018	.0300	.1605	.5037
-.100	.0019	.0343	.1800	.5330
-.050	.0028	.0431	.2055	.5578
0.000	.0048	.0571	.2352	.5915
.050	.0087	.0732	.2619	.6222
.100	.0089	.0797	.2847	.6517
.150	.0109	.0933	.3167	.6800
.200	.0154	.1135	.3517	.7036
.250	.0238	.1428	.3919	.7394
.300	.0424	.1790	.4289	.7703
.350	.0431	.1912	.4583	.7996
.400	.0471	.2097	.4930	.8293
.450	.0577	.2426	.5400	.8565
.500	.0772	.2837	.5862	.8795
.550	.0921	.3309	.6367	.9012
.600	.2038	.4343	.6941	.9183
.650	.2077	.4514	.7200	.9332
.700	.2190	.4857	.7568	.9479
.750	.2356	.5195	.7898	.9629
.800	.2617	.5702	.8345	.9759
.850	.3696	.6683	.8833	.9852
.900	.3885	.7117	.9159	.9926
.950	.5070	.8111	.9590	.9978
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .80000000 ALPHA2= .60000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	0.0000	.0004	.0390
-.900	0.0000	0.0000	.0018	.0779
-.850	0.0000	0.0000	.0046	.1169
-.800	0.0000	.0001	.0089	.1558
-.750	0.0000	.0003	.0149	.1948
-.700	0.0000	.0006	.0226	.2336
-.650	0.0000	.0012	.0322	.2730
-.600	0.0000	.0021	.0437	.3110
-.550	0.0000	.0034	.0572	.3513
-.500	0.0000	.0052	.0729	.3887
-.450	.0001	.0077	.0904	.4292
-.400	.0002	.0110	.1107	.4683
-.350	.0003	.0155	.1328	.5039
-.300	.0006	.0207	.1570	.5473
-.250	.0009	.0274	.1844	.5854
-.200	.0015	.0365	.2139	.6194
-.150	.0024	.0462	.2448	.6632
-.100	.0032	.0579	.2795	.7041
-.050	.0047	.0724	.3170	.7405
0.000	.0073	.0912	.3565	.7742
.050	.0112	.1104	.3966	.8053
.100	.0141	.1322	.4398	.8344
.150	.0188	.1581	.4842	.8602
.200	.0256	.1889	.5298	.8823
.250	.0365	.2265	.5750	.9028
.300	.0539	.2638	.6180	.9209
.350	.0610	.2991	.6621	.9385
.400	.0808	.3472	.7063	.9524
.450	.0983	.3924	.7482	.9640
.500	.1261	.4466	.7892	.9733
.550	.1574	.5087	.8270	.9805
.600	.2467	.5720	.8589	.9866
.650	.2690	.6172	.8882	.9913
.700	.3009	.6679	.9157	.9947
.750	.3746	.7312	.9395	.9969
.800	.4216	.7799	.9591	.9985
.850	.4996	.8387	.9758	.9994
.900	.5777	.8914	.9882	.9998
.950	.6795	.9456	.9966	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .80000000 ALPHA2= .80000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	0.0000	.0010	.0676
-.900	0.0000	0.0000	.0048	.1353
-.850	0.0000	.0001	.0122	.2029
-.800	0.0000	.0004	.0236	.2706
-.750	0.0000	.0010	.0393	.3382
-.700	0.0000	.0022	.0597	.4055
-.650	0.0000	.0042	.0850	.4745
-.600	0.0000	.0072	.1154	.5394
-.550	.0001	.0117	.1511	.6109
-.500	.0002	.0180	.1923	.6743
-.450	.0003	.0266	.2385	.7296
-.400	.0006	.0381	.2888	.7784
-.350	.0011	.0527	.3423	.8211
-.300	.0019	.0711	.3981	.8571
-.250	.0032	.0936	.4551	.8884
-.200	.0051	.1205	.5125	.9150
-.150	.0079	.1519	.5693	.9359
-.100	.0118	.1881	.6244	.9524
-.050	.0172	.2290	.6769	.9652
0.000	.0249	.2741	.7259	.9751
.050	.0348	.3231	.7710	.9828
.100	.0476	.3756	.8119	.9882
.150	.0641	.4307	.8481	.9921
.200	.0850	.4875	.8795	.9949
.250	.1116	.5449	.9064	.9968
.300	.1429	.6019	.9289	.9981
.350	.1789	.6577	.9473	.9989
.400	.2216	.7112	.9619	.9994
.450	.2704	.7615	.9734	.9997
.500	.3257	.8077	.9820	.9998
.550	.3891	.8489	.9883	.9999
.600	.4606	.8846	.9928	1.0000
.650	.5255	.9150	.9958	1.0000
.700	.5945	.9403	.9978	1.0000
.750	.6618	.9607	.9990	1.0000
.800	.7294	.9764	.9996	1.0000
.850	.7971	.9878	.9999	1.0000
.900	.8647	.9952	1.0000	1.0000
.950	.9324	.9990	1.0000	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .60000000   ALPHA2= .40000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	.0011	.0250	.2087
-.900	0.0000	.0040	.0500	.2815
-.850	.0002	.0087	.0750	.3326
-.800	.0004	.0133	.1000	.3873
-.750	.0010	.0216	.1250	.4158
-.700	.0016	.0280	.1500	.4670
-.650	.0022	.0365	.1750	.4899
-.600	.0037	.0484	.2000	.5056
-.550	.0080	.0646	.2250	.5514
-.500	.0081	.0699	.2500	.5837
-.450	.0094	.0813	.2750	.6036
-.400	.0119	.0965	.3000	.6164
-.350	.0152	.1120	.3250	.6322
-.300	.0267	.1406	.3500	.6367
-.250	.0400	.1615	.3750	.6893
-.200	.0403	.1683	.4000	.7153
-.150	.0414	.1791	.4250	.7359
-.100	.0464	.1988	.4500	.7503
-.050	.0485	.2132	.4750	.7621
0.000	.0596	.2413	.5000	.7705
.050	.0727	.2639	.5250	.7854
.100	.0784	.2892	.5500	.7920
.150	.1004	.3255	.5750	.7979
.200	.2000	.4000	.6000	.8000
.250	.2001	.4037	.6250	.8616
.300	.2008	.4130	.6500	.8832
.350	.2022	.4252	.6750	.8993
.400	.2065	.4419	.7000	.9167
.450	.2104	.4620	.7250	.9250
.500	.2320	.4969	.7500	.9379
.550	.2351	.5132	.7750	.9487
.600	.2476	.5448	.8000	.9541
.650	.2669	.5826	.8250	.9591
.700	.3601	.6422	.8500	.9723
.750	.3633	.6598	.8750	.9817
.800	.3856	.6981	.9000	.9876
.850	.4018	.7371	.9250	.9912
.900	.4907	.7959	.9500	.9963
.950	.5748	.8618	.9750	.9988
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .60000000 ALPHA2= .60000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	.0016	.0324	.2419
-.900	.0001	.0056	.0648	.3259
-.850	.0003	.0118	.0969	.3843
-.800	.0006	.0188	.1300	.4498
-.750	.0014	.0295	.1615	.4803
-.700	.0021	.0384	.1900	.5275
-.650	.0033	.0519	.2301	.5772
-.600	.0055	.0682	.2616	.5979
-.550	.0095	.0842	.2842	.6095
-.500	.0107	.0961	.3166	.6593
-.450	.0145	.1165	.3584	.7034
-.400	.0180	.1366	.3951	.7281
-.350	.0242	.1610	.4252	.7418
-.300	.0350	.1889	.4559	.7573
-.250	.0476	.2105	.4737	.7619
-.200	.0505	.2256	.4977	.7780
-.150	.0557	.2489	.5438	.8471
-.100	.0682	.2817	.5855	.8729
-.050	.0810	.3100	.6204	.8906
0.000	.0898	.3415	.6585	.9102
.050	.1094	.3796	.6900	.9190
.100	.1271	.4144	.7183	.9316
.150	.1514	.4562	.7510	.9443
.200	.2219	.5023	.7744	.9495
.250	.2381	.5263	.7895	.9524
.300	.2427	.5441	.8111	.9648
.350	.2581	.5748	.8390	.9758
.400	.2718	.6049	.8634	.9820
.450	.2967	.6415	.8835	.9854
.500	.3398	.6834	.9039	.9893
.550	.3905	.7158	.9158	.9905
.600	.4021	.7384	.9318	.9945
.650	.4228	.7699	.9481	.9967
.700	.4719	.8101	.9616	.9979
.750	.5197	.8385	.9705	.9986
.800	.5500	.8700	.9812	.9994
.850	.6157	.9031	.9882	.9997
.900	.6740	.9352	.9944	.9999
.950	.7525	.9676	.9984	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .60000000 ALPHA2= .80000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	0.0000	.0034	.0547	.3126
-.900	.0002	.0118	.1086	.4221
-.850	.0006	.0242	.1613	.4996
-.800	.0015	.0409	.2201	.5788
-.750	.0031	.0605	.2689	.6245
-.700	.0053	.0843	.3319	.7009
-.650	.0087	.1118	.3828	.7308
-.600	.0134	.1411	.4280	.7535
-.550	.0195	.1731	.4916	.8393
-.500	.0267	.2107	.5532	.8761
-.450	.0360	.2518	.6076	.9017
-.400	.0476	.2937	.6529	.9190
-.350	.0615	.3379	.7008	.9393
-.300	.0791	.3820	.7361	.9461
-.250	.0972	.4250	.7736	.9629
-.200	.1177	.4702	.8110	.9749
-.150	.1398	.5158	.8419	.9812
-.100	.1656	.5602	.8678	.9859
-.050	.1947	.6034	.8896	.9889
0.000	.2258	.6435	.9089	.9926
.050	.2595	.6830	.9275	.9954
.100	.2959	.7206	.9421	.9968
.150	.3368	.7552	.9538	.9976
.200	.3806	.7861	.9635	.9985
.250	.4146	.8156	.9726	.9992
.300	.4527	.8430	.9793	.9994
.350	.4961	.8672	.9845	.9997
.400	.5317	.8893	.9890	.9998
.450	.5708	.9096	.9923	.9999
.500	.6113	.9271	.9948	1.0000
.550	.6487	.9428	.9966	1.0000
.600	.6890	.9563	.9979	1.0000
.650	.7270	.9678	.9988	1.0000
.700	.7664	.9774	.9994	1.0000
.750	.8052	.9851	.9997	1.0000
.800	.8442	.9911	.9999	1.0000
.850	.8831	.9954	1.0000	1.0000
.900	.9221	.9982	1.0000	1.0000
.950	.9610	.9996	1.0000	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .40000000 ALPHA2= .60000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	.0012	.0250	.1409	.4451
-.900	.0037	.0500	.2041	.5093
-.850	.0088	.0750	.2627	.5974
-.800	.0125	.1000	.3019	.6144
-.750	.0183	.1250	.3401	.6366
-.700	.0280	.1500	.3578	.6399
-.650	.0409	.1750	.4170	.7315
-.600	.0459	.2000	.4552	.7522
-.550	.0513	.2250	.4868	.7649
-.500	.0624	.2500	.5031	.7680
-.450	.0750	.2750	.5380	.7896
-.400	.0832	.3000	.5581	.7935
-.350	.1007	.3250	.5748	.7978
-.300	.1169	.3500	.5870	.7992
-.250	.1398	.3750	.5963	.7999
-.200	.2000	.4000	.6000	.8000
-.150	.2019	.4250	.6731	.8961
-.100	.2080	.4500	.7108	.9214
-.050	.2146	.4750	.7361	.9273
0.000	.2295	.5000	.7586	.9402
.050	.2379	.5250	.7867	.9515
.100	.2499	.5500	.8012	.9536
.150	.2641	.5750	.8209	.9586
.200	.2847	.6000	.8317	.9597
.250	.3118	.6250	.8385	.9600
.300	.3631	.6500	.8590	.9727
.350	.3678	.6750	.8880	.9849
.400	.3836	.7000	.9034	.9880
.450	.3965	.7250	.9187	.9906
.500	.4162	.7500	.9301	.9919
.550	.4495	.7750	.9354	.9920
.600	.4943	.8000	.9515	.9963
.650	.5102	.8250	.9635	.9978
.700	.5330	.8500	.9720	.9984
.750	.5844	.8750	.9783	.9990
.800	.6128	.9000	.9867	.9996
.850	.6676	.9250	.9913	.9998
.900	.7186	.9500	.9960	1.0000
.950	.7863	.9750	.9989	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .40000000 ALPHA2= .80000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	.0022	.0400	.1862	.4970
-.900	.0074	.0841	.2883	.6115
-.850	.0148	.1167	.3317	.6304
-.800	.0241	.1649	.4286	.7391
-.750	.0371	.2102	.4805	.7644
-.700	.0521	.2434	.5145	.7809
-.650	.0668	.2800	.5486	.7923
-.600	.0817	.3061	.5658	.7962
-.550	.0989	.3643	.6701	.9075
-.500	.1205	.4123	.7144	.9239
-.450	.1435	.4596	.7570	.9425
-.400	.1707	.5071	.7903	.9528
-.350	.2004	.5418	.8089	.9569
-.300	.2297	.5711	.8210	.9576
-.250	.2606	.6086	.8576	.9761
-.200	.2964	.6474	.8858	.9848
-.150	.3200	.6832	.9066	.9892
-.100	.3483	.7154	.9204	.9911
-.050	.3778	.7381	.9268	.9913
0.000	.4085	.7652	.9430	.9952
.050	.4422	.7940	.9567	.9972
.100	.4670	.8201	.9657	.9981
.150	.4963	.8395	.9700	.9982
.200	.5268	.8591	.9772	.9990
.250	.5590	.8805	.9837	.9995
.300	.5850	.8981	.9872	.9996
.350	.6152	.9125	.9901	.9998
.400	.6472	.9283	.9935	.9999
.450	.6738	.9411	.9952	.9999
.500	.7051	.9523	.9968	1.0000
.550	.7337	.9629	.9979	1.0000
.600	.7641	.9714	.9987	1.0000
.650	.7930	.9791	.9993	1.0000
.700	.8229	.9852	.9996	1.0000
.750	.8523	.9903	.9998	1.0000
.800	.8819	.9942	.9999	1.0000
.850	.9114	.9970	1.0000	1.0000
.900	.9409	.9988	1.0000	1.0000
.950	.9705	.9998	1.0000	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

ALPHA1= .20000000 ALPHA2= .80000000

X	PI1= .200	PI1= .400	PI1= .600	PI1= .800
-1.000	0.0000	0.0000	0.0000	0.0000
-.950	.0250	.1455	.3560	.6398
-.900	.0500	.2196	.4832	.7651
-.850	.0750	.2738	.5327	.7847
-.800	.1000	.3243	.5731	.7969
-.750	.1250	.3636	.5933	.7998
-.700	.1500	.3849	.5985	.8000
-.650	.1750	.3969	.5999	.8000
-.600	.2000	.4000	.6000	.8000
-.550	.2250	.5000	.7595	.9373
-.500	.2500	.5489	.8054	.9564
-.450	.2750	.5874	.8258	.9589
-.400	.3000	.6182	.8373	.9600
-.350	.3250	.6333	.8397	.9600
-.300	.3500	.6358	.8360	.9583
-.250	.3750	.6844	.8879	.9808
-.200	.4000	.7293	.9221	.9913
-.150	.4250	.7577	.9318	.9919
-.100	.4500	.7765	.9358	.9920
-.050	.4750	.7799	.9337	.9915
0.000	.5000	.8107	.9551	.9962
.050	.5250	.8426	.9705	.9984
.100	.5500	.8612	.9738	.9984
.150	.5750	.8669	.9733	.9983
.200	.6000	.8864	.9821	.9992
.250	.6250	.9091	.9888	.9997
.300	.6500	.9184	.9893	.9997
.350	.6750	.9285	.9918	.9998
.400	.7000	.9455	.9955	.9999
.450	.7250	.9524	.9959	.9999
.500	.7500	.9623	.9975	1.0000
.550	.7750	.9705	.9983	1.0000
.600	.8000	.9774	.9990	1.0000
.650	.8250	.9834	.9994	1.0000
.700	.8500	.9883	.9997	1.0000
.750	.8750	.9923	.9999	1.0000
.800	.9000	.9954	.9999	1.0000
.850	.9250	.9976	1.0000	1.0000
.900	.9500	.9991	1.0000	1.0000
.950	.9750	.9998	1.0000	1.0000
1.000	1.0000	1.0000	1.0000	1.0000

TABLE 4.

CHAPTER V

NUMERICAL TABULATION OF THE TRANSFORMED LIMITING DISTRIBUTIONS OF THE SUBJECT CONTROLLED MODEL

§ 5.1 Introduction

Because of the difficulties involved in the Subject-Controlled Model, numerical solutions may be the only possible method of finding the forms of the limiting distributions. Exact solutions of the known purely singular cases of $\alpha_1 + \alpha_2 < 1$ cannot be constructed in general, and no absolutely continuous distributions have been found.

However, the functional integral equations satisfied by the limiting distributions may be solved numerically to obtain approximations of the limiting distribution functions for all values of the parameters. The accuracy of the method cannot be theoretically determined, but in the centro-symmetric cases the numerical solutions appear to be sufficiently accurate for practical purposes. Comparing the numerical solutions with Monte-Carlo approximations also indicates that the numerical tabulation is reliable and it can be used to obtain greater precision than is feasible by the Monte-Carlo method.

§5.2 Numerical Procedure

Consider the problem of calculating the values of the distribution functions from the functional integral equations on a tabular set of points $\{x_i\}$ lying in the domain $(-1,1)$. These points are chosen such that $x_i = ih-1$ $(i = 0, 1, 2, \ldots, 2N)$ where $h = 1/N$.

On this tabular set, the functional integral equations may be written

$$(5.1)\qquad -F(x_i)+[\lambda_1+(x_i+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F[(x_i+1-\alpha_1)/\alpha_1]$$
$$+[1-\lambda_2+(1-x_i)(\lambda_2-\lambda_1)/(2\alpha_2)]F[(x_i-1+\alpha_2)/\alpha_2]-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{(x_i-1+\alpha_2)/\alpha_2}^{(x_i+1-\alpha_1)/\alpha_1} F(u)du = 0,$$

$$(\lambda_2 > \lambda_1),$$

and

$$(5.2)\qquad -F(x_i)+[\lambda_1+(x_i-1)(\lambda_1-\lambda_2)/(2\alpha_1)]F[(x_i-1+\alpha_1)/\alpha_1]$$
$$+[1-\lambda_2-(x_i-1)(\lambda_1-\lambda_2)/(2\alpha_2)]F[(x_i+1-\alpha_2)/\alpha_2]+\tfrac{1}{2}(\lambda_1-\lambda_2)\int_{(x_i-1+\alpha_1)/\alpha_1}^{(x_i+1-\alpha_2)/\alpha_2} F(u)du = 0,$$

$$(\lambda_1 > \lambda_2,\ \lambda_1 \neq 1,\ \lambda_2 \neq 0),$$

where $(i = 1, 2, 3, \ldots, 2N-1)$.

Imposing the lower boundary condition of $F(x) = 0, \quad x \leq -1$,

the above equations become

$$(5.3)\qquad -F(x_i)+[\lambda_1+(x_i+1)(\lambda_2-\lambda_1)/(2\alpha_1)]F[(x_i+1-\alpha_1)/\alpha_1]-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{-1}^{(x_i+1-\alpha_1)/\alpha_1} F(u)du = 0,$$

$$(i = 1, 2, 3, \ldots m),$$

where $\quad mh-1 \leq 1-2\alpha_2, \quad (m+1)h-1 \quad 1-2\alpha_2,$

and

$$(5.4)\qquad -F(x_i)+[1-\lambda_2-(x_i+1)(\lambda_1-\lambda_2)/(2\alpha_2)]F[(x_i+1-\alpha_2)/\alpha_2]+\tfrac{1}{2}(\lambda_1-\lambda_2)\int_{-1}^{(x_i+1-\alpha_2)/\alpha_2} F(u)du = 0,$$

$$(i = 1, 2, 3, \ldots, m),$$

where $mh-1 \leq 1-2\alpha_1$, $(m+1)h-1 > 1-2\alpha_1$.

Imposing the upper boundary condition gives

$$(5.5) \qquad -F(x_i)+\lambda_2+[1-\lambda_2-(1-x_i)(\lambda_2-\lambda_1)/(2\alpha_2)]F[(x_i-1+\alpha_2)/\alpha_2]$$

$$-\tfrac{1}{2}(\lambda_2-\lambda_1)\int_{(x_i-1+\alpha_2)/\alpha_2}^{1} F(u)du = 0 \quad , \qquad (i = n,\ n+1,\ \ldots,\ 2N-1) \ ,$$

where $(n-1)h-1< 2\alpha_1-1$, $nh-1 \geq 2\alpha_1-1$,

and

$$(5.6) \qquad -F(x_i)+1-\lambda_1+[\lambda_1+(x_i-1)(\lambda_1-\lambda_2)/(2\alpha_1)]F[(x_i-1+\alpha_1)/\alpha_1]$$

$$+\tfrac{1}{2}(\lambda_1-\lambda_2)\int_{(x_i-1+\alpha_1)/\alpha_1}^{1} F(u)du = 0 \quad , \qquad (i=n,n+1,n+2,\ldots,2N-1) \ ;$$

where $(n-1)h-1 < 2\alpha_2-1$, $nh-1 \geq 2\alpha_2-1$.

As in Chapter IV, the arguments $(x_i+1-\alpha_1)/\alpha_1$, $(x_i-1+\alpha_2)/\alpha_2$, $(x_i+1-\alpha_2)/\alpha_2$, $(x_i-1+\alpha_1)/\alpha_1$ will not, in general, lie on the tabular points $\{x_i\}$. Thus equations (5.1), (5.3), (5.5) and (5.2), (5.4), (5.6) define systems of $2N-1$ equations in $6N-m-n-3$ unknown values of F as well as $2N-1$ undetermined integrals. The values of the distribution function at off-tabular points may be expressed in terms of its values on adjacent tabular points by the Lagrange Interpolation formula described in Chapter IV.

The integrals in each of the equations (5.1)-(5.6) can be approximated numerically by weighted sums of $F(x_i)$, $F[(x_i+1-\alpha')/\alpha']$

and $F[(x_i-1+\alpha^*)/\alpha^*]$, where α' and α^* are to be interpreted as shown on page 71 . Wherever possible, a Simpson's Rule approximation was used, but many situations arose where the simpler Trapezoidal sum was required. The following notation enables us to discuss a number of cases together.

Let the largest argument be $(x+1-\alpha')/\alpha'$ and the smallest be $(x-1+\alpha^*)/\alpha^*$. For $x = x_i$, let $(x_i+1-\alpha')/\alpha'$ be denoted by A_i and $(x_i-1+\alpha^*)/\alpha^*$ be denoted by C_i .

Suppose $x_{k-1} < A_i \leq x_k$ and $x_{j-1} < c_i \leq x_j$

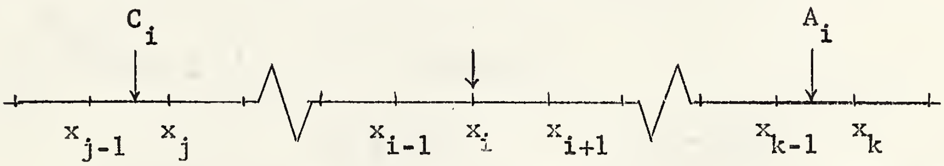

Figure 9: Arrangement of the arguments $A_i = (x_i+1-\alpha')/\alpha'$ and $C_i = (x_i-1+\alpha^*)/\alpha^*$ when $x = x_i$

The integral to be approximated is of the form

$$\int_{C_i}^{A_i} F(u)\, du \quad . \tag{5.7}$$

Each of the following cases requires special consideration:

a) $A_i \neq x_k$

i) $k-j < 2$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_i)]+\tfrac{1}{2}(A_i - x_i)[F(x_i)+F(A_i)]$$

ii) $k-j = 2$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_j)]+\tfrac{1}{2}h[F(x_j)+F(x_{j+1})]$$
$$+\tfrac{1}{2}(A_i - x_{k-1})[F(x_{k-1})+F(A_i)] \; .$$

iii) $k-j = 2r-1$, $r = 2, 3, 4, \ldots$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_j)]$$
$$+(h/3)\left[F(x_j)+4\sum_{\ell=1}^{(k-j-1)/2} F(x_{j+2\ell-1}) + 2\sum_{\ell=1}^{(h-j-3)/2} F(x_{j+2\ell})+F(x_{k-1})\right]$$
$$+ \tfrac{1}{2}(A_i - x_{k-1})[F(x_{k-1})+F(A_i)] \; .$$

iv) $k-j = 2r$, $r = 2, 3, 4, \ldots$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_j)]+\tfrac{1}{2}h[F(x_j)+F(x_{j+1})]$$
$$+(h/3\left[F(x_{j+1})+4\sum_{\ell=1}^{(k-j-2)/2} F(x_{j+1+2\ell})+2\sum_{\ell=1}^{(k-j-4)/2} F(x_{j+2+2\ell})+F(x_{k-1})\right]$$
$$+\tfrac{1}{2}h[F(x_{k-1})+F(A_i)].$$

Note: when $C_i \leq -1$, $j = 1$ and equation (5.3) or (5.4) is used, when $A_i \geq 1$, $k = 2N$ and equation (5.5) or (5.6) is used, and when $C_i = x_j$ the first Trapezoidal Sum is vacuous.

b) $A_i = x_k$

i) $k-j < 2$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_i - C_i)[F(C_i)+F(x_i)] + \tfrac{1}{2}(h)[F(x_i)+F(x_{i+1})]$$

ii) $k-j=2$

$$\int_{C_i}^{A_i} F(u)du = \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_j)]+(h/3)[F(x_j)+4F(x_{j+1})+F(x_{j+2})]$$

iii) $k-j=2r-1 \quad r = 2,3,4,\ldots$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_j)]+\tfrac{1}{2}h[F(x_j)+F(x_{j+1})]$$

$$(h/3)\left[F(x_{j+1})+ 4 \sum_{\ell=1}^{\frac{1}{2}(k-j-1)} F(x_{j+2})+ 2 \sum_{\ell=1}^{\frac{1}{2}(k-j-3)} F(x_{j+1+2\ell}) + F(x_k)\right].$$

iv) $k-j=2r \quad r = 2,3,\ldots$

$$\int_{C_i}^{A_i} F(u)du \simeq \tfrac{1}{2}(x_j - C_i)[F(C_i)+F(x_j)]$$

$$+(h/3)\left[F(x_j)+ 4 \sum_{\ell=0}^{\frac{1}{2}(k-j-2)} F(x_{j+1+2}) + 2 \sum_{\ell=0}^{\frac{1}{2}(k-j-u)} F(x_{j+2+2\ell})+F(x_k)\right].$$

When $C_i \leq -1$, $j=1$ and (5.3) or (5.4) is used,

when $A_i \geq 1$, $k=2N$ and (5.5) or (5.6) is used,

when $C_i = x_j$, the first trapezoidal sum becomes vacuuous.

In all cases, when C_i or A_i does not lie on a tabular point, $F(C_i)$ and/or $F(A_i)$ is expressed in terms of the Lagrange Interpolation polynomial.

The ALGOL form of the program for setting up the system of linear equations follows.

```
procedure        Subject-Controlled Distribution Functions;
array            A[1:50,1:51],  x[0:51],  y[1:50],  C[1:4];
real             h, alfa1, alfa2, lam1, lam2, arg1, arg2, coef1, coef2, coef3,
                 swtch, a11, a12, B, sgn, sgn1, hf, ht, d, e , comax, comin, arg,
                 cof, hp, hpstor;
integer          i, j, k, max, max2, max1, lead, init, jstor, kick, no;
Boolean          kode, jump, mop, leap;
begin comment    This procedure tabulates the values of the limiting distribu-
                 tion functions by calculating the numerical solutions of the
                 functional integral equations satisfied by the limiting dis-
                 tributions of transformed response probabilities;
                 read (h, alfa1, alfa2, lam1, lam2);
                 max:= 2.0/h;   max 2:= max+1;
                 for  i:=0 step 1 until max 2 do x[i]:=i × h - 1.0;
                 if  lam1 > lam2 then go to type 2 else go to type 1;
type 1:          kode:=true; arg1:=1.0-alfa1; arg2:=alfa2-one;
                 coef1:=lam2-lam1; coef2:=coef1/(2.0× alfa1);
                 coef3:=coef1/(2.0×alfa2);
                 swtch:=1.0-2.0×alfa2;  a11:=alfa1, a12:=alfa2;  B:=-lam2;
                 sgn:= -1.00 ;
                 go to calculate;
type 2:          kode:=false; arg1:=1.0-alfa2; arg2:=alfa1-1.0; coef1:=lam1-lam2;
                 coef2:=coef1/(2.0 ×alfa2); coef3:=coef1/(2.0×alfa1);
                 swtch:=1.0-2.0×alfa1; a11:=alfa2; a12:=alfa1; B:=lam1-1.0;
                 sgn:=1.0 ;
```

```
comment        The above program segment reads in the parameters and initial-
               izes for the required functional integral equation;
calculate:     max1:=max-1; hf:=2.0×h×coef1×sgn/3.000;
               ht:=h×coef1×sgn/3.000;  sgn1:=sgn; leap:=true;
               d:=h×sgn×coef1×.25000;  e:=sgn×coef1×.250000;
               for  i=1  step 1 until max 1 do
begin comment  This loop sets up the ith row of the matrix  A[i,j];
               for  j:=1 step 1 until max do  A[i,j]:=0.0;
               A[i,1]:=-1.0
               if kode then begin comax:=(x[i]+1.0)×coef2+lam1;
                                  comin:=(1.0-x[i])×coef3+1.0-lam2;
                             end
               else begin         comax:=1.0-lam2-(x[i]+1.0)×coef2;
                                  comin:=(x[i]-1.0)×coef3+lam1;
                             end
               if leap then go to pace else go to run;
               pace: arg:=(x[i]+arg1)/a11; coef:=comax;
               if arg < 1.0 then go to march else go to trot;
               march: for j:=1, j+1, while  x[j] < arg do;
               if  x[j] = arg  then go to equal else go to unequal;
               equal: hp:=0.0;  A[i,j]:=A[i,j]+cof; go to test;
               unequal: if jump then hp:=arg-x[j] else hp:=x[j]-arg;
               if j ≤ 3 then lead:=0 else
               if j ≥ max1 then lead:=max-4 else lead:=j-4;
               Interp (lead, max, arg,x,C);
comment        Interp is the Lagrange Interpolation routine described in
               Chapter IV;
```

```
            for  k:=1  step 1 until 4 do A[i,k+lead]:=A[i,k+lead]+c(k)×(coef+hp×e);
            go to test;
trot:       leap:=false;    sgn1:=-sgn;
run:        hpstor:=0.0;    A[i,max]:=A[i,max]+B;  jstor:=max;
            go to skip;
test:       if jump then begin jstor:=j; hpstor:=hp; go to skip; end
                   else go to set;
skip:       if  x[i] > swtch then begin cof:=comin; arg:=(x[i]+arg2)/a12;
            jump:=false;  go to unequal; end
            else begin
                A[i,jstor-1]:=A[i,jstor-1]+hpstor×e; l:=jstor;
                if hpstor=0 then go to tabpt else go to nontabpt;
                end
tabpt:      if l/2.0-entier (l/2.0)=0 then begin
                kick:=1; no:=jstor-1;  go to integral end
            else begin  A[i,l]:=A[i,l]+d; kick:=2; no:=jstor-2 go to stintegral
                end
nontabpt:   if l/2.0-entier (l/2.0)=0 then begin
                A[i,l]:=A[i,l]+d; kick:=2; no:=jstor-2; go to stintegral;
                end
            else begin kick:=1; no:=jstor-2; go to integral; end
set:        A[i,jstor-1]:=A[i,jstor-1]+Hpstor×e; A[i,j]:=A[i,j]+hp×e;
            l:=jstor-j;
            if hpstor=0 then go to tabpt 2 else
            begin if  l < 2  then go to main else
              if  l = 2  then go to place else
              begin if  l/2.0-entier (l/2.0)=0 then
```

```
                begin kick:=j+2;  A[i,j]:=A[i,j]+d;
                      A[i,j+1]:=A[i,j+1]+d; no:=jstor-2;
                      go to stintegral;
                end
           else begin kick:=j+1;  no:=jstor-2; go to integral; end
        end
end
tabpt 2:   if  1 < 2  then go to place  else
           begin if  1=2 then begin kick:=j+1; no:=jstor-1;
                                  go to stintegral; end
                              else if 1/2.0-entier(1/2.0)=0 then
                                  begin kick:=j+1; no:=jstor-1;
                                        go to stintegral;
                                  end
                              else begin kick:=j+2; A[i,j]:=A[i,j]+d;
                                      A[i,j+1]:=A[i,j+1]+d; no:=jstor-1;
                                      go to stintegral;
                                  end
place:     A[i,j]:=A[i,j]+d; A[i,j+1]:=A[i,j+1]+sgn1×h×coef1×.25000;
           go to main;
stintegral: A[i,kick-1]=A[i,kick-1]+sgn×h×coef1/6.0000;
integral:  mop:=true;
           for k:=kick step 1 until no do
           begin if mop then begin A[i,k]:=A[i,k]+hf; mop:=false; end
                       else begin A[i,k]:=A[i,k]+ht; mop:=true; end
           end
           A[i,no]:=A[i,no]+sgn1×h×coef1/6.000;
```

```
main:      end  This ends the loop for setting up the matrix equation;
           GAUSEL (maxl, A, y); comment  This statement calls a Gaussian
                                   elimination with interchanges routine to
                                   solve the matrix equation;
           Punch (laml, alfal);  Punch (lam2, alfa2);  Punch (-1,0);
           for  i=1 step 1 until maxl do  Punch (x[i],y[i]);
           Punch (1,1);
           end  Subject-Controlled Distribution Functions;
```

§5.3 Discussion of the Numerical Results.

The procedure described above was programmed in the Fortran II language and several examples were run on the University of Alberta's IBM 1620 computer. Hardware floating point with a seven decimal digit mantissa and two decimal digit exponent was used. The parameter values chosen were the ones used by Bush and Mosteller in calculating approximate distributions by Monte-Carlo Methods.

Useful error bounds cannot be obtained in the Subject case since practically nothing is known of the properties of the limiting distributions. We cannot even compare the numerical values with known solutions. However, there is one check that indicates the numerical procedure is reliable. This is given by the centro-symmetric distributions of $\alpha_1=\alpha_2$, $\lambda_1+\lambda_2=1$ ([1]p107). Two of the numerical examples satisfy this condition and with a tabular interval $h=.05$, the error in centro-symmetry is $< 2\times10^{-6}$. (see Table 5)

Comparison of the results of the Monte-Carlo approximations with the numerical solutions of the functional integral equations indicates that the Monte-Carlo results tend to be much smoother. Since the Monte-Carlo

approximations were determined by at most four thousand trials, the maximum accuracy possible is one to two decimal digits. On the other hand, the numerical solution of the functional integral equations seems to yield an accuracy of four to five digits.

LAMBDA1= 0.00000
LAMBDA2= 1.00000

ALPHA1= .500000
ALPHA2= .500000

X	F
-1.0000	0.0000
-.9500	.0001
-.9000	.0013
-.8500	.0044
-.8000	.0164
-.7500	.0249
-.7000	.0404
-.6500	.0841
-.6000	.1102
-.5500	.1431
-.5000	.1500
-.4500	.1581
-.4000	.2040
-.3500	.2478
-.3000	.3364
-.2500	.3749
-.2000	.4062
-.1500	.4641
-.1000	.4850
-.0500	.4987
0.0000	.5000
.0500	.5013
.1000	.5150
.1500	.5359
.2000	.5938
.2500	.6251
.3000	.6636
.3500	.7522
.4000	.7960
.4500	.8419
.5000	.8500
.5500	.8569
.6000	.8898
.6500	.9159
.7000	.9596
.7500	.9751
.8000	.9836
.8500	.9956
.9000	.9987
.9500	.9999
1.0000	1.0000

TABLE 5. NUMERICAL APPROXIMATION OF SOME LIMITING DISTRIBUTIONS OF THE SUBJECT-CONTROLLED MODEL FOR SELECTED VALUES OF THE PARAMETERS

LAMBDA1= .80000
LAMBDA2= .20000

ALPHA1= .500000
ALPHA2= .500000

X	F
-1.0000	0.0000
-.9500	.1484
-.9000	.1877
-.8500	.2219
-.8000	.2395
-.7500	.2644
-.7000	.2866
-.6500	.2974
-.6000	.3120
-.5500	.3232
-.5000	.3496
-.4500	.3750
-.4000	.3845
-.3500	.3952
-.3000	.4022
-.2500	.4147
-.2000	.4275
-.1500	.4353
-.1000	.4482
-.0500	.4607
0.0000	.5000
.0500	.5393
.1000	.5518
.1500	.5647
.2000	.5725
.2500	.5852
.3000	.5978
.3500	.6048
.4000	.6155
.4500	.6250
.5000	.6504
.5500	.6768
.6000	.6880
.6500	.7026
.7000	.7133
.7500	.7356
.8000	.7605
.8500	.7781
.9000	.8123
.9500	.8516
1.0000	1.0000

TABLE 5.

LAMBDA1= .75000
LAMBDA2= .10000

ALPHA1= .600000
ALPHA2= .900000

X	F
-1.0000	0.0000
-.9500	.0001
-.9000	.0001
-.8500	.0002
-.8000	.0003
-.7500	.0004
-.7000	.0005
-.6500	.0007
-.6000	.0009
-.5500	.0012
-.5000	.0015
-.4500	.0018
-.4000	.0023
-.3500	.0029
-.3000	.0036
-.2500	.0045
-.2000	.0055
-.1500	.0069
-.1000	.0085
-.0500	.0106
0.0000	.0132
.0500	.0164
.1000	.0203
.1500	.0254
.2000	.0316
.2500	.0390
.3000	.0495
.3500	.0613
.4000	.0759
.4500	.0972
.5000	.1200
.5500	.1479
.6000	.1917
.6500	.2355
.7000	.2898
.7500	.3566
.8000	.4699
.8500	.5384
.9000	.6465
.9500	.7629
1.0000	1.0000

TABLE 5.

LAMBDA1= .75000
LAMBDA2= .20000

ALPHA1= .600000
ALPHA2= .700000

X	F
-1.0000	0.0000
-.9500	.0363
-.9000	.0557
-.8500	.0724
-.8000	.0887
-.7500	.1061
-.7000	.1203
-.6500	.1390
-.6000	.1514
-.5500	.1699
-.5000	.1883
-.4500	.2000
-.4000	.2214
-.3500	.2409
-.3000	.2645
-.2500	.2764
-.2000	.2941
-.1500	.3225
-.1000	.3458
-.0500	.3735
0.0000	.4024
.0500	.4203
.1000	.4424
.1500	.4689
.2000	.4919
.2500	.5164
.3000	.5435
.3500	.5595
.4000	.6230
.4500	.6387
.5000	.6602
.5500	.6813
.6000	.7075
.6500	.7459
.7000	.7641
.7500	.7875
.8000	.8232
.8500	.8464
.9000	.8817
.9500	.9203
1.0000	1.0000

TABLE 5.

CHAPTER VI

CONCLUSION

The functional equations satisfied by the limiting distribution functions of the Bush and Mosteller stochastic learning models have yielded much valuable information about the limiting distribution of response probabilities. Exact solutions for the distribution functions can be derived from these equations in many cases and numerical solutions of the functional equations appear to be a fast and accurate method of tabulating the limiting distributions for all values of the parameters.

The distribution functional equation derived from the Experimenter-Controlled Model has been particularly valuable. For all $\alpha_1+\alpha_2 \leq 1$, it can be used to construct the limiting distribution function by a very simple mechanical algorithm. For $\alpha_1+\alpha_2 > 1$, the functional equation yields a construction for absolutely continuous distribution functions where they exist. By pursuing this approach further, more precise criteria for absolute continuity might be derived. However, even where analytic methods fail, numerical solutions appear to provide sufficient information to be of value for applied problems.

The distribution functional integral equations of the Subject-Controlled Model have not yet proved to be of particular value. For $\alpha_1+\alpha_2 < 1$, the functional equations merely demonstrate the form of the limiting distributions. However, future work may yield a method of constructing exact solutions from these apparently recalcitrant equations. As for the distribution functions corresponding to values of $\alpha_1+\alpha_2 > 1$, almost no useful information has

been derived. Despite the poverty of analytical results, the functional integral equations do appear to be amenable to numerical methods. Even when $\alpha_1 \neq \alpha_2$ (where uniqueness of the solutions has not been established) the numerical results seem to be stable. This tends to imply that the solutions are unique.

Until more is learned about the structure of these types of functional equations, we may have to depend upon numerical results. It may even be hoped that numerical work can serve as a guide to theoreticians in this field.

BIBLIOGRAPHY

[1] Bush, R.R. and Mosteller, F: Stochastic Models for Learning. Wiley (1955).

[2] McGregor, J.R. and Hui, Y.Y: "Limiting Distributions Associated With Certain Stochastic Learning Models". Annals of Mathematical Statistics 33, (1962).

[3] Karlin, S: "Some Random Walks Arising in Learning Models I". Pacific Journal of Mathematics, 3, (1953).

[4] Kendall, M.G. and Stuart, A: The Advanced Theory of Statistics, Volume I. London: C. Griffin and Co. (1958).

[5] Jessen, B. and Wintner, A: "Distribution Functions and the Riemann Zeta Function". Transactions of the American Mathematical Society 38, (1935).

[6] Kershner, R. and Wintner, A: "On a Family of Symmetric Bernoulli Convolutions." American Journal of Mathematics, 57, (1935).

[7] Kemeny, and Snell: Markoff Processes and Learning Theory". Psychometrika, 22, (1957).

[8] Wintner, A: "On Convergent Poisson Processes." American Journal of Mathematics 57, (1935).

[9] Zidek, J: Asymptotic Distributions of Response Probabilities for a Stochastic Learning Model". Unpublished M.Sc. Thesis, University of Alberta.

[10] Erdos, P: "On a Family of Symmetric Bernoulli Convolutions." American Journal of Mathematics, 61, (1939).

[11] Lukacs, E: Characteristic Functions, Charles Griffin and Co. Ltd. (1960).

[12] Bush, R.R. and Mosteller, F: "A Stochastic Model With Applications to Learning." Annals of Mathematical Statistics, 24, (1953).